

Higher Spin Double Field Theory : A Proposal

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Abstract

We construct a double field theory coupled to the fields present in Vasiliev's equations. Employing the “semi-covariant” differential geometry, we spell a functional in which each term is completely covariant with respect to $\mathbf{O}(4, 4)$ T-duality, doubled diffeomorphisms, $\mathbf{Spin}(1, 3)$ local Lorentz symmetry and, separately, $\mathbf{HS}(4)$ higher spin gauge symmetry. We identify a minimal set of BPS-like conditions whose solutions automatically satisfy the full Euler-Lagrange equations. As such a solution, we derive a linear dilaton vacuum. With extra algebraic constraints further supplemented, the BPS-like conditions reduce to the bosonic Vasiliev equations.

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1 Introduction

Both higher spin gravity and double field theory extend Einstein’s gravity beyond the Riemannian paradigm. The idea of unifying the graviton either with an infinite tower of massless fields of ever higher spin, or with other massless NS-NS fields, appears to require us to extend Riemannian geometry in both cases.

While free Higher Spin (HS) theories were well explored, notably by Fronsdal [1, 2], it was Vasiliev who first managed to write down a set of gauge invariant equations which describe the full interactions among the infinite tower of higher spin fields [3] (see also [4–9] for reviews). The higher spin gauge symmetries are expected to be so huge as to prevent quantum corrections and thereby ensure ultraviolet finiteness of higher spin gravity at quantum level (see *e.g.* [10] for an early statement of this conjecture).

Moreover, the higher spin symmetry enhancement of string theory in the ultra-Planckian regime has been argued to be responsible for the remarkable softness of its scattering amplitudes [11]. The holographic duality provides a complementary perspective on these expectations. On the one hand, interactions of massless higher spin particles suffer from serious restrictions in flat spacetime (see *e.g.* [12, 13] for reviews on no-go theorems) and, in fact, Vasiliev equations require a nonvanishing cosmological constant and possess (anti) de Sitter spacetime as the most symmetric background. On the other hand, the holographic dictionary suggests that higher spin gravity theories around anti de Sitter spacetime (*AdS*) are dual to free or integrable conformal field theories on the boundary (see *e.g.* [14–16] for some reviews on higher spin holography). Since what replaces the S-matrix in *AdS* are boundary correlators, the remarkable simplicity of such holograms (free or integrable CFTs) is the avatar of the absence of quantum corrections and the extreme softness of higher spin gravity. While the holographic duality suggests a relation between higher spin gravity and *closed* string theory, there exists a striking resemblance between the classification of allowed internal symmetry group extensions in Vasiliev theory and Chan-Paton factors in *open* string theory. Accordingly, the Vasiliev system has also been argued (see *e.g.* the review [17] and references therein) to yield an effective description for the first Regge trajectory of the *open* string in the tensionless limit. Let us emphasize that, in the latter interpretation, the Vasiliev multiplet appears as some sort of ‘matter’ sector and it becomes questionable whether the massless spin-two particles in the Vasiliev multiplet should be treated as the graviton.

Double Field Theory (DFT) is an extension of the Einstein gravity in accordance with, especially, string theory [18–23]. The primary goal of DFT was to manifest T-duality, or to reformulate supergravity in a way that the hidden $O(D, D)$ structure becomes manifest (see *e.g.* [24–26] for reviews). As the $O(D, D)$ T-duality rotations mix the Riemannian metric, the Kalb-Ramond B -field and the scalar dilaton, the geometrization of the whole massless NS-NS sector was inevitable in DFT, as once anticipated by Siegel [19]. This is clearly in contrast to the conventional picture of (super)gravity, where the Riemannian metric is regarded as the only geometric object while the B -field and the scalar dilaton are viewed as ‘matter’ living on the background the metric characterizes.

Concretely in [27, 28], the DFT generalization of the Christoffel symbols was derived which is compatible with, and hence comprised of, the whole massless NS-NS sector. It defines so-called the ‘semi-covariant’ derivative and curvature which can be completely covariantized [27–31]. This line of geometrical development, often called the ‘semi-covariant geometry’, has turned out to have the potential to replace the conventional Riemannian geometry, while it manifests, for every single term in formulas, $O(D, D)$ T-duality, twofold local Lorentz symmetries, *i.e.* $\mathbf{Spin}(1, D-1) \times \mathbf{Spin}(D-1, 1)$, and doubled diffeomorphisms (DFT-diffeomorphisms) which unifies the ordinary Riemannian diffeomorphisms and the B -field gauge symmetry (*c.f.* ‘Generalized Geometry’ [32–34]). For example, for $D=10$, the half-maximal as well as the maximal supersymmetric double field theories have been constructed to the full order in fermions [35, 36], of which the latter unifies IIA and IIB supergravities since the twofold spin groups remove intrinsically the relative chirality difference of IIA and IIB. Further, as for $D=4$, it has been shown possible to double field theorize the Standard Model, without any extra physical degree

introduced [37]: equipped with the semi-covariant geometry, it can couple to an arbitrary NS-NS background in a completely covariant manner. In particular, the incorporation of the twofold spin groups, $\mathbf{Spin}(1,3) \times \mathbf{Spin}(3,1)$, into the Standard Model lead to an experimentally testable prediction that the quarks and the leptons may belong to the different spin classes. The semi-covariant differential geometry also facilitates efficient perturbation analyses as well as Wald type Noether charge derivations in double field theories [38–40].¹

It is the purpose of the present paper to apply the semi-covariant geometry and propose a “higher spin double field theory”. To be more precise, we extend DFT by introducing the fields present in Vasiliev equations which we treat as ‘matter’ minimally coupled to the massless NS-NS sector, *i.e.* the geometric objects in DFT.² Accordingly, in contrast to the pure Vasiliev equations, we shall have the notion of covariant derivatives which accompany the DFT-Christoffel connection as well as the local DFT-Lorentz spin connection. For concreteness, putting $D = 4$, we restrict ourselves to the spacetime dimension four. We focus on one of the twofold spin groups in DFT and consider its extension to include the higher spin gauge symmetry, $\mathbf{HS}(4)$,

$$\mathbf{Spin}(1,3) \longrightarrow \mathbf{Spin}(1,3) \times \mathbf{HS}(4). \quad (1.1)$$

The algebra denoted here $\mathbf{HS}(4)$ is the algebra of gauge symmetries of Vasiliev equations.³ In our proposal, $\mathbf{Spin}(1,3)$ and $\mathbf{HS}(4)$ are realized independently, although they share the same spinorial indices. This is analogous to the case of DFT where the T-duality and doubled diffeomorphisms act on the same indices, while they differ from each other.

We present a functional and derive the corresponding Euler-Lagrange equations. Every term in our formulas is going to be completely covariant under $\mathbf{O}(4,4)$ T-duality, DFT-diffeomorphisms, $\mathbf{Spin}(1,3)$ local Lorentz symmetry and higher spin gauge symmetry, $\mathbf{HS}(4)$. We identify a minimal set of BPS-like conditions which are strong enough to solve automatically the full equations of motion. We also derive a vacuum configuration as a solution to the BPS-like conditions, which is characterized by a linear DFT-dilaton and a constant or flat DFT-vielbein. Finally, we discuss a consistent truncation of the BPS-like conditions to the bosonic Vasiliev equations after imposing extra algebraic conditions. We also comment on a possible alternative without any extra condition being imposed, in view of the open string theory interpretation of Ref. [17].

¹For other approaches, we refer to *e.g.* [41–44].

²In our proposal, DFT is the genuine gravity theory (closed string related) while the HS sector is treated as matter (possibly open string related). The geometrization of the entire higher spin degrees of freedom is of course desirable but it goes beyond the scope of the present work.

³This algebra was referred to as “embedding algebra” in [45]. It must be distinguished from what is usually referred to as “higher spin algebra” and sometimes denoted as $\mathfrak{hs}(4)$ in the literature. In fact, as is well known, the latter algebra is just a subalgebra of the former, $\mathfrak{hs}(4) \subset \mathbf{HS}(4)$, because it only depends on half of the oscillators. Nevertheless, in the present paper we will often loosely refer to $\mathbf{HS}(4)$ as the higher spin gauge algebra because here we never particularize to its subalgebra, $\mathfrak{hs}(4)$.

Let us clarify why, as the attentive reader may have noticed, in the abstract and introduction, we have refrained from calling the invariant functional which we propose, an “action”. This terminological caution aims at avoiding possible misunderstandings since no standard action principle is presently known for the precise Vasiliev equations.⁴ The functional we propose is merely an extension of DFT where two fields taking values in the $\mathbf{HS}(4)$ algebra, which can be off-shell identified as the fields present in Vasiliev equations, are added and treated as some sort of ‘matter’ minimally coupled to the pure DFT (and hence possibly the open string interpretation). In fact, we want to stress that the equations of motion we obtain from the functional are *not* by themselves equal to the Vasiliev equations. Nevertheless, the latter can be obtained as a consistent truncation of the former. To be more precise, the differential equations (the flatness and covariant constancy) in Vasiliev system of equations are identified in the present paper as part of the BPS-like conditions, while the algebraic equations, including the so-called deformed oscillator algebra, in Vasiliev system appear as a subclass of the solutions to the remaining BPS-like conditions. They are not the most general solutions and hence, strictly speaking, must be supplemented by hand if we want to precisely recover the full Vasiliev equations.⁵ This word of caution being said, the BPS truncation of the Euler-Lagrange equations of our proposed functional provides a suggestive double field theory covariantization of the Vasiliev bosonic equations.⁶

The rest of the paper is organized as follows.

- In section 2, we summarize our proposal of Higher Spin Double Field Theory, which decomposes into two parts: kinematics and dynamics.
- Section 3 delivers detailed exposition. In a self-contained manner, we review $\mathbf{Spin}(1, 3)$ Clifford algebra (gamma matrices, Majorana condition), the Wick (*i.e.* normal) ordered star product, and the semi-covariant geometry of double field theory including its complete covariantization. We detail the derivation of the Euler-Lagrange equations as well as the BPS-like conditions. We discuss the linear dilaton vacuum solution, and explain the consistent truncation of the BPS-like conditions to the bosonic Vasiliev equations.
- We conclude with comments in section 4.

⁴However, a nonstandard proposal is available (*c.f.* the review [46] and references therein for details). See also [47] for a proposal of an on-shell action and related discussion.

⁵Off-shell, the two $\mathbf{HS}(4)$ -valued fields are simply a gauge potential and a bosonic spinor field taking values in the infinite-dimensional algebra. Once the BPS-like conditions are imposed, then the gauge potential is flat (up to projection) and the spinor field is covariantly constant. Strictly speaking, it is only when the extra algebraic equations are eventually imposed that they can be interpreted as the Vasiliev’s HS gauge fields.

⁶Since the Vasiliev fields are here treated as “matter” minimally coupled to DFT, the massless spin-two field in the HS tower, if any, should be distinguished from the metric in the DFT multiplet. When the Vasiliev theory is thought as arising from the first Regge trajectory of open string in some tensionless limit (a point of view followed here), then the massless spin-two particle in the higher spin multiplet should indeed *not* be identified with the graviton. For this reason, we avoided the use of the term “higher spin gravity” when referring to our proposal.

2 Proposal

In this section, we depict the salient features of the Higher Spin Double Field Theory (HS-DFT) we propose. We first state the kinematic ingredients, such as the symmetries, coordinates and the field content. We then introduce the master derivative, in terms of which we spell the proposed action, its full Euler-Lagrange equations and the BPS-like conditions. We sketch a linear dilaton vacuum solution, and prescribes the consistent truncation of the BPS-like equations to the undoubled Vasiliev equations. The detailed exposition will follow in the next section 3.

2.1 Kinematics of HS-DFT

- First of all, the symmetries of the proposed HS-DFT are the following.
 - $\mathbf{O}(4, 4)$ *T-duality*,
 - *DFT-diffeomorphisms*,
 - $\mathbf{Spin}(1, 3)$ *local Lorentz symmetry*,
 - $\mathbf{HS}(4)$ *higher spin gauge symmetry*.
- The relevant indices are summarized in Table 1 as for our convention.

Index	Representation	Range
A, B, \dots	$\mathbf{O}(4, 4)$ vector	$1, 2, \dots 8$
	DFT-diffeomorphisms	
α, β, \dots	$\mathbf{Spin}(1, 3)$ spinor	$1, 2, 3, 4$
	$\mathbf{HS}(4)$ spinor	
p, q, \dots	$\mathbf{Spin}(1, 3)$ vector	$0, 1, 2, 3$

Table 1: Index for each symmetry representation. As is always the case with DFT, $\mathbf{O}(4, 4)$ and DFT-diffeomorphisms share the same vectorial indices, *i.e.* the capital Latin alphabet letters. Similarly, $\mathbf{Spin}(1, 3)$ and $\mathbf{HS}(4)$ use the same spinorial indices, *i.e.* the small greek letters. In our proposal of HS-DFT, $\mathbf{Spin}(1, 3)$ is not a subgroup of $\mathbf{HS}(4)$, *c.f.* (2.15) and (2.17).

In particular, the $\mathbf{O}(4, 4)$ group is characterized by the constant invariant metric, \mathcal{J}_{AB} , put in an off block diagonal form,

$$L_A{}^B \in \mathbf{O}(4, 4) \quad : \quad L_A{}^C L_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB}, \quad \mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.1)$$

The capital Latin alphabet indices are used commonly for both $\mathbf{O}(4, 4)$ and DFT-diffeomorphisms, and they can be freely lowered or raised by the above invariant metric, or its inverse, \mathcal{J}^{AB} .

The $\mathbf{Spin}(1, 3)$ vectorial, small Latin alphabet indices are subject to the mostly positive Minkowskian metric, η_{pq} , which also sets the Clifford algebra of the 4×4 gamma matrices, $(\gamma^p)^\alpha{}_\beta$,

$$\gamma^p \gamma^q + \gamma^q \gamma^p = 2\eta^{pq}, \quad \eta_{pq} = \text{diag}(- + + +), \quad \gamma^{(5)} = (\gamma^{(5)})^\dagger = (\gamma^{(5)})^{-1} = -i\gamma^{0123}. \quad (2.2)$$

We set a unitary and Hermitian matrix, $\mathbf{A} = \mathbf{A}^\dagger = \mathbf{A}^{-1} \equiv -i\gamma^0$, satisfying for $n = 0, 1, 2, 3, 4$,

$$(\gamma^p)^\dagger = \gamma_p = -\mathbf{A} \gamma^p \mathbf{A}^{-1}, \quad (\mathbf{A} \gamma^{p_1 p_2 \dots p_n})^\dagger = (-1)^{n(n+1)/2} \mathbf{A} \gamma^{p_1 p_2 \dots p_n}, \quad \gamma^{(5)} = -\mathbf{A} \gamma^{(5)} \mathbf{A}^{-1}. \quad (2.3)$$

The charge conjugation matrix is skew-symmetric and unitary, $\mathbf{C} = -\mathbf{C}^T = (\mathbf{C}^\dagger)^{-1}$, satisfying

$$(\gamma^p)^T = -\mathbf{C} \gamma^p \mathbf{C}^{-1}, \quad (\mathbf{C} \gamma^{p_1 p_2 \dots p_n})_{\alpha\beta} = -(-1)^{n(n+1)/2} (\mathbf{C} \gamma^{p_1 p_2 \dots p_n})_{\beta\alpha}, \quad (\gamma^{(5)})^T = \mathbf{C} \gamma^{(5)} \mathbf{C}^{-1}. \quad (2.4)$$

Especially, $\mathbf{C} \gamma^p$ and $\mathbf{C} \gamma^{pq}$ are symmetric to fulfill the completeness relation,

$$\frac{1}{2}(\delta_\alpha{}^\gamma \delta_\beta{}^\delta + \delta_\beta{}^\gamma \delta_\alpha{}^\delta) = \frac{1}{4}(\mathbf{C} \gamma_p)_{\alpha\beta} (\gamma^p \mathbf{C}^{-1})^{\gamma\delta} - \frac{1}{8}(\mathbf{C} \gamma_{pq})_{\alpha\beta} (\gamma^{pq} \mathbf{C}^{-1})^{\gamma\delta}. \quad (2.5)$$

On the other hand, \mathbf{C} , $\mathbf{C} \gamma^{(5)}$ and $\mathbf{C} \gamma^{(5)} \gamma^p$ are skew-symmetric, with the completeness relation,

$$\frac{1}{2}(\delta_\alpha{}^\gamma \delta_\beta{}^\delta - \delta_\beta{}^\gamma \delta_\alpha{}^\delta) = -\frac{1}{4} \mathbf{C}_{\alpha\beta} \mathbf{C}^{-1\gamma\delta} - \frac{1}{4}(\mathbf{C} \gamma^{(5)})_{\alpha\beta} (\gamma^{(5)} \mathbf{C}^{-1})^{\gamma\delta} - \frac{1}{4}(\mathbf{C} \gamma^{(5)} \gamma^p)_{\alpha\beta} (\gamma_p \gamma^{(5)} \mathbf{C}^{-1})^{\gamma\delta}. \quad (2.6)$$

- The four-dimensional spacetime is described by adopting the eight-dimensional *doubled-yet-gauged* coordinate system, $\{x^A\}$, $A = 1, 2, \dots, 8$ [48, 49]. This means that – as reviewed in section 3.1 – all the variables, *e.g.* fields, local parameters, their arbitrary derivatives and products, are subject to the *section condition* [23],

$$\partial_A \partial^A = 0. \quad (2.7)$$

Consequently, the theory is confined to live on a four-dimensional ‘section’. The diffeomorphisms in such a doubled-yet-gauged coordinate system are generated by the generalized Lie derivative used in DFT, *c.f.* (2.14).

- The higher spin gauge symmetry, $\mathbf{HS}(4)$, is realized through a star product defined over an internal space with $\mathbf{Spin}(1, 3)$ spinorial coordinates, ζ^α and $\bar{\zeta}_\beta$,

$$[\zeta^\alpha, \bar{\zeta}_\beta]_\star = \delta^\alpha_\beta, \quad [\zeta^\alpha, \zeta^\beta]_\star = 0, \quad [\bar{\zeta}_\alpha, \bar{\zeta}_\beta]_\star = 0. \quad (2.8)$$

In particular, the spinorial coordinates are bosonic (*i.e.* even Grassmannian) Dirac spinors having four complex components, and $\bar{\zeta}_\alpha$ is the Dirac conjugate of ζ^α , *i.e.* $\bar{\zeta} \equiv \zeta^\dagger \mathbf{A}$.

As is the case with Vasiliev [3], the star product represents a non-commutative algebra which is not Weyl but Wick ordered, and hence generically it reads

$$f(x, \zeta, \bar{\zeta}) \star g(x, \zeta, \bar{\zeta}) = f(x, \zeta, \bar{\zeta}) \exp \left(\frac{\overleftarrow{\partial}}{\partial \zeta^\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha} \right) g(x, \zeta, \bar{\zeta}). \quad (2.9)$$

This presentation of the algebra of gauge symmetries in Vasiliev equations is not standard. The detailed relation to the more traditional Y and Z oscillators is given in section 3.8.

- *The field content* of HS-DFT is

$$d(x), \quad V_{Ap}(x), \quad \mathcal{W}_A(x, \zeta, \bar{\zeta}), \quad \Psi^\alpha(x, \zeta, \bar{\zeta}), \quad (2.10)$$

which correspond to a DFT-dilaton, a DFT-vielbein, a higher spin gauge potential and a bosonic $\mathbf{Spin}(1, 3)$ Majorana spinor. In particular, the DFT-dilaton gives rise to a scalar density with weight one after exponentiation, e^{-2d} , and the DFT-vielbein is subject to one defining property,

$$V_{Ap} V^A_q = \eta_{pq}. \quad (2.11)$$

Consequently, it generates a pair of symmetric and orthogonal projectors,

$$\begin{aligned} P_{AB} &= P_{BA} := V_{Ap} V_B^p, & \bar{P}_{AB} &= \bar{P}_{BA} := \mathcal{J}_{AB} - P_{AB}, \\ P_A^B P_B^C &= P_A^C, & \bar{P}_A^B \bar{P}_B^C &= \bar{P}_A^C, & P_A^B \bar{P}_B^C &= 0. \end{aligned} \quad (2.12)$$

The DFT-vielbein may convert the $\mathbf{O}(4, 4)$ indices to $\mathbf{Spin}(1, 3)$ vector indices and *vice versa*, if there is no room for ambiguity, *e.g.* $\partial_p = V^A_p \partial_A$, $\gamma^A = V^A_p \gamma^p$, but $V_A^p \partial_p = P_A^B \partial_B \neq \partial_A$.

The higher spin gauge potential is anti-Hermitian, $\mathcal{W}_A = -(\mathcal{W}_A)^\dagger$, and the Majorana property of the bosonic spinor, Ψ^α , means that its Dirac conjugate equals the charge conjugate, $\bar{\Psi} = \Psi^\dagger \mathbf{A} = \Psi^T \mathbf{C}$.

It is crucial for us to postulate that – as the arguments indicate – $d(x)$ and $V_{Ap}(x)$ are independent of the internal spinorial coordinates, and hence they are HS singlets, *c.f.* (2.16). On the other hand, $\mathcal{W}_A(x, \zeta, \bar{\zeta})$ and $\Psi^\alpha(x, \zeta, \bar{\zeta})$ are HS algebra valued: they depend on the internal coordinates and thus transform under the HS gauge symmetry in a nontrivial manner, *c.f.* (2.15).

- *Symmetry transformation rules.*

- The $\mathbf{O}(4, 4)$ T-duality is a global symmetry rotating the uppercase alphabet indices of not only the DFT-vielbein, V_{Ap} , and the HS gauge potential, \mathcal{W}_A , but also the coordinates themselves, x^A , which are the common arguments in all the fields in (2.10): with $L \in \mathbf{O}(4, 4)$, the finite transformations are given by

$$\begin{pmatrix} x^A \\ d(x) \\ V_{Ap}(x) \\ \mathcal{W}_A(x, \zeta, \bar{\zeta}) \\ \Psi^\alpha(x, \zeta, \bar{\zeta}) \end{pmatrix} \longrightarrow \begin{pmatrix} x'^A = x^B L_B^A, \\ d(x') \\ L_A^B V_{Bp}(x') \\ L_A^B \mathcal{W}_B(x', \zeta, \bar{\zeta}) \\ \Psi^\alpha(x', \zeta, \bar{\zeta}) \end{pmatrix}. \quad (2.13)$$

- The diffeomorphisms on the doubled-yet-gauged coordinate system, *i.e.* *DFT-diffeomorphisms*, are generated by the generalized Lie derivative [19, 51], such that

$$\begin{aligned} \delta_X d &= X^A \partial_A d - \frac{1}{2} \partial_A X^A, & \delta_X V_{Ap} &= X^B \partial_B V_{Ap} + (\partial_A X^B - \partial^B X_A) V_{Bp}, \\ \delta_X \Psi^\alpha &= X^A \partial_A \Psi^\alpha, & \delta_X \mathcal{W}_A &= X^B \partial_B \mathcal{W}_A + (\partial_A X^B - \partial^B X_A) \mathcal{W}_B, \end{aligned} \quad (2.14)$$

where $X^A(x)$ is an $\mathbf{O}(4, 4)$ vectorial diffeomorphism parameter which should be independent of the internal spinorial coordinates, $\zeta, \bar{\zeta}$.

- The higher spin gauge symmetry, $\mathbf{HS}(4)$, is realized through the adjoint action of the star product,

$$\delta_{\mathcal{T}} \Psi = [\mathcal{T}, \Psi]_\star, \quad \delta_{\mathcal{T}} \mathcal{W}_A = -(\partial_A \mathcal{T} + [\mathcal{W}_A, \mathcal{T}]_\star) \equiv -\mathcal{D}_A \mathcal{T}, \quad (2.15)$$

where $\mathcal{T}(x, \zeta, \bar{\zeta})$ is a local parameter of the HS gauge symmetry, which is anti-Hermitian, $\mathcal{T} = -(\mathcal{T})^\dagger$, and depends arbitrarily on all the coordinates. The DFT-dilaton and the DFT-vielbein are independent of the internal spinorial coordinates, and hence they are higher spin gauge singlets,

$$\delta_{\mathcal{T}} d = [\mathcal{T}, d(x)]_\star = 0, \quad \delta_{\mathcal{T}} V_{Ap} = [\mathcal{T}, V_{Ap}(x)]_\star = 0. \quad (2.16)$$

- The **Spin**(1, 3) local Lorentz symmetry rotates the explicit unbarred lowercase indices, specifically the vectorial small roman letter of V_{Ap} and the spinorial greek letter of Ψ^α : with a skew-symmetric, arbitrarily x -dependent local parameter, $\omega_{pq}(x) = -\omega_{qp}(x)$, its infinitesimal transformations are given by

$$\begin{aligned}\delta_\omega d(x) &= 0, & \delta_\omega V_{Ap}(x) &= \omega_p{}^q(x) V_{Aq}(x), \\ \delta_\omega \mathcal{W}_A(x, \zeta, \bar{\zeta}) &= 0, & \delta_\omega \Psi^\alpha(x, \zeta, \bar{\zeta}) &= \frac{1}{4} \omega_{pq}(x) (\gamma^{pq})^\alpha{}_\beta \Psi^\beta(x, \zeta, \bar{\zeta}),\end{aligned}\tag{2.17}$$

where *a priori* the internal spinorial coordinates, $\zeta^\alpha, \bar{\zeta}_\beta$, inside $\mathcal{W}_A(x, \zeta, \bar{\zeta})$ and $\Psi^\alpha(x, \zeta, \bar{\zeta})$ are **not** rotated. However, the above local Lorentz transformations can be modified to include the higher spin gauge symmetry with a particular form of the local parameter,

$$\mathcal{T}_\omega(x, z) \equiv \frac{1}{4} \omega_{pq}(x) \bar{\zeta} \gamma^{pq} \zeta.\tag{2.18}$$

From

$$[\mathcal{T}_\omega, \zeta^\alpha]_\star = -\frac{1}{4} \omega_{pq}(x) (\gamma^{pq} \zeta)^\alpha, \quad [\mathcal{T}_\omega, \bar{\zeta}_\alpha]_\star = +\frac{1}{4} \omega_{pq}(x) (\bar{\zeta} \gamma^{pq})_\alpha,\tag{2.19}$$

the modified **Spin**(1, 3) local Lorentz transformation rule, $\delta_\omega \rightarrow \delta_{\omega+\mathcal{T}_\omega}$, then rotates not only the explicit **Spin**(1, 3) indices of the fields but also all the internal spinorial coordinates: while the HS gauge singlet fields transform in the same way as before (2.17),

$$\delta_{\omega+\mathcal{T}_\omega} d = 0, \quad \delta_{\omega+\mathcal{T}_\omega} V_{Ap} = \omega_p{}^q V_{Aq},\tag{2.20}$$

the HS gauge adjoint fields transform in a novel way compared to (2.17),

$$\begin{aligned}\delta_{\omega+\mathcal{T}_\omega} \mathcal{W}_A &= -\partial_A \mathcal{T}_\omega + [\mathcal{T}_\omega, \mathcal{W}_A]_\star \\ &= -\frac{1}{4} \partial_A \omega_{pq} \bar{\zeta} \gamma^{pq} \zeta - \frac{1}{4} \omega_{pq} (\gamma^{pq} \zeta)^\beta \partial_\beta \mathcal{W}_A + \frac{1}{4} \omega_{pq} (\bar{\zeta} \gamma^{pq})_\beta \bar{\partial}^\beta \mathcal{W}_A, \\ \delta_{\omega+\mathcal{T}_\omega} \Psi^\alpha &= \frac{1}{4} \omega_{pq} (\gamma^{pq})^\alpha{}_\beta \Psi^\beta + [\mathcal{T}_\omega, \Psi]_\star \\ &= \frac{1}{4} \omega_{pq} (\gamma^{pq})^\alpha{}_\beta \Psi^\beta - \frac{1}{4} \omega_{pq} (\gamma^{pq} \zeta)^\beta \partial_\beta \Psi^\alpha + \frac{1}{4} \omega_{pq} (\bar{\zeta} \gamma^{pq})_\beta \bar{\partial}^\beta \Psi^\alpha,\end{aligned}\tag{2.21}$$

where we set $\partial_\beta \equiv \frac{\partial}{\partial \zeta^\beta}$ and $\bar{\partial}^\beta \equiv \frac{\partial}{\partial \bar{\zeta}_\beta}$, and clearly all the spinors are to be rotated.

In either case of (2.17) or (2.21), since the higher spin gauge symmetry does not rotate any of the explicit, external indices of V_{Ap} and Ψ^α (2.15), (2.16), the local Lorentz symmetry, T-duality and the DFT-diffeomorphisms are all intrinsically different from the higher spin gauge symmetry.

- We define *the master derivative* of HS-DFT,

$$\mathcal{D}_A = \partial_A + \Gamma_A(x) + \Phi_A(x) + \mathcal{W}_A(x, \zeta, \bar{\zeta}), \quad (2.22)$$

which takes care of the DFT-diffeomorphisms (2.14), the **Spin**(1, 3) local Lorentz symmetry (2.17), and the **HS**(4) gauge symmetry (2.15), by employing the relevant three connections:

- (i) the DFT extension of the Christoffel connection [28] (*c.f.* [27]),

$$\begin{aligned} \Gamma_{CAB} := & 2(P\partial_C P\bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E) \partial_D P_{EC} \\ & - \frac{4}{3}(\bar{P}_{C[A} \bar{P}_{B]}{}^D + P_{C[A} P_{B]}{}^D)(\partial_D d + (P\partial^E P\bar{P})_{[ED]}) , \end{aligned} \quad (2.23)$$

- (ii) the local Lorentz spin connection [35, 36],

$$\Phi_{Apq} := V^B{}_p(\partial_A V_{Bq} + \Gamma_{AB}{}^C V_{Cq}) = V^B{}_p \nabla_A V_{Bq}, \quad (2.24)$$

- (iii) the higher spin gauge potential, \mathcal{W}_A , in the adjoint representation of the star product.

In (2.24), ∇_A denotes the ‘semi-covariant derivative’ set by the DFT-Christoffel connection [28],

$$\nabla_A := \partial_A + \Gamma_A, \quad (2.25)$$

which satisfies, among others (*c.f.* section 3.4),

$$\nabla_A \mathcal{J}_{BC} = 0, \quad \nabla_A P_{BC} = 0, \quad \nabla_A \bar{P}_{BC} = 0, \quad \nabla_A d = 0. \quad (2.26)$$

As an example, note the expression,

$$\mathcal{D}_A \Psi^\alpha = \partial_A \Psi^\alpha + \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\alpha{}_\beta \Psi^\beta + [\mathcal{W}_A, \Psi^\alpha]_\star, \quad (2.27)$$

and also the fact that the expression (2.24) follows from the compatibility of the master derivative with the DFT-vielbein which is HS singlet,

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = \nabla_A V_{Bp} + \Phi_{Ap}{}^q V_{Bq} = 0. \quad (2.28)$$

By appropriate contractions with the projectors (2.12) or the DFT-vielbein, the master derivative can be completely covariantized, *e.g.* (2.43) – (2.46).

2.2 Dynamics of HS-DFT

- The proposed functional, *i.e.* action, of HS-DFT is

$$\mathcal{S}_{\text{HS-DFT}} = \int d^4x \mathcal{L}_{\text{HS-DFT}}, \quad \mathcal{L}_{\text{HS-DFT}} = \mathcal{L}_{\text{DFT}} + \mathcal{L}_{\text{HS}}, \quad (2.29)$$

where the x -integral is to be taken over a four-dimensional section of choice, and the HS-DFT Lagrangian consists of two parts:

- (i) the ‘pure’ DFT Lagrangian,

$$\mathcal{L}_{\text{DFT}} = e^{-2d} \left[(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} - 2\Lambda_{\text{DFT}} \right], \quad (2.30)$$

- (ii) the ‘matter’ HS Lagrangian,

$$\mathcal{L}_{\text{HS}} = g_{\text{HS}}^{-2} e^{-2d} \text{Tr} \left[P^{AC} \bar{P}^{BD} \mathcal{F}_{AB}^{\mathcal{W}} \star \mathcal{F}_{CD}^{\mathcal{W}} + \bar{\Psi} \star \gamma^{(5)} \gamma^A \mathcal{D}_A \Psi - V_{\star}(\Psi) \right]. \quad (2.31)$$

The ‘pure’ DFT Lagrangian (2.30) contains the DFT version of the cosmological constant, Λ_{DFT} , of which the value can be arbitrary, and the ‘semi-covariant’ four-index curvature [28],

$$S_{ABCD} := \frac{1}{2} (R_{ABCD} + R_{CDAB} - \Gamma_{AB}^E \Gamma_{ECD}), \quad (2.32)$$

with

$$R_{CDAB} := \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}. \quad (2.33)$$

In the ‘matter’ HS Lagrangian (2.31), g_{HS} denotes a coupling constant; $V_{\star}(\Psi)$ is a star-producted scalar potential of the bosonic spinor, Ψ ; and the trace means the formal $(\zeta, \bar{\zeta})$ -integrals over the (real eight-dimensional) internal space,

$$\text{Tr} \left[\begin{array}{c} \cdot \\ \cdot \end{array} \right] := \int d^4\zeta \int d^4\bar{\zeta} \left[\begin{array}{c} \cdot \\ \cdot \end{array} \right]. \quad (2.34)$$

We also set the ‘semi-covariant’ field strength of the higher spin gauge potential, following [29, 37],

$$\mathcal{F}_{AB}^{\mathcal{W}} := \nabla_A \mathcal{W}_B - \nabla_B \mathcal{W}_A + [\mathcal{W}_A, \mathcal{W}_B]_{\star}. \quad (2.35)$$

Since the Majorana spinor, Ψ , is bosonic while $\mathbf{C}\gamma^{(5)}\gamma^A$ is skew-symmetric, the kinetic term of Ψ , *i.e.* the second term in (2.31), is not a total derivative. On account of the cosmological constant, Λ_{DFT} , without loss of generality, we assume the ‘absolute’ minimum of the potential to vanish, *i.e.* $\min[V_{\star}(\Psi)] = 0$.

We emphasize that each term in the Lagrangian is completely covariant, under all the symmetries: $\mathbf{O}(4, 4)$ T-duality, DFT-diffeomorphisms, $\mathbf{Spin}(1, 3)$ local Lorentz symmetry and $\mathbf{HS}(4)$ higher spin gauge symmetry.

- The Euler-Lagrange equations of the full action (2.29) are as follows.

The equation of motion of the DFT-dilaton implies that the on-shell Lagrangian should vanish,

$$\mathcal{L}_{\text{HS-DFT}} = 0. \quad (2.36)$$

For the DFT-vielbein, with $S_{AB} := S_{ACB}{}^C$, we have both

$$\bar{\Psi} \star \gamma^{pq} \gamma^{(5)} \gamma^A \mathcal{D}_A \Psi = 0, \quad (2.37)$$

and, as a HS-DFT extension of the Einstein equation,

$$P_A{}^C \bar{P}_B{}^D \left(S_{CD} + \frac{1}{2} g_{\text{HS}}^{-2} \text{Tr} \left[\{ \mathcal{F}^{\mathcal{W}} \star (P - \bar{P}) \mathcal{F}^{\mathcal{W}} \}_{CD} + \nabla_E (\mathcal{F}_{CD}^{\mathcal{W}} \star \mathcal{W}^E) + \frac{1}{2} \bar{\Psi} \star \gamma^{(5)} \gamma_C \mathcal{D}_D \Psi \right] \right) = 0. \quad (2.38)$$

The equation of motion of the higher spin gauge potential is also twofold, as it implies both

$$\mathcal{D}_B (P \mathcal{F}^{\mathcal{W}} \bar{P})^{BA} = 0, \quad (2.39)$$

and

$$\mathcal{D}_B (P \mathcal{F}^{\mathcal{W}} \bar{P})^{AB} - \frac{1}{2} [\Psi^\alpha, \Psi^\beta]_\star (\mathbf{C} \gamma^{(5)} \gamma^A)_{\alpha\beta} = 0. \quad (2.40)$$

Finally, for the bosonic Majorana spinor, we have a HS-DFT extension of the Dirac equation,

$$\gamma^A \mathcal{D}_A \Psi - \frac{1}{2} \gamma^{(5)} \mathbf{C}^{-1} \partial_\Psi \text{Tr} [V_\star(\Psi)] = 0, \quad (2.41)$$

which actually implies (2.37) provided the potential, $V_\star(\Psi)$, is **Spin**(1, 3) singlet.

- The full set of the Euler-Lagrange equations (2.36)–(2.41), are automatically fulfilled, provided the following ‘stronger’ equations, or *BPS-like conditions*, hold:

$$(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} - 2\Lambda_{\text{DFT}} - g_{\text{HS}}^{-2} \text{Tr} [V_\star(\Psi)] = 0, \quad (2.42)$$

$$P_A{}^C \bar{P}_B{}^D S_{CD} = 0, \quad (2.43)$$

and

$$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}^{\mathcal{W}} = 0, \quad (2.44)$$

$$\bar{P}_A{}^B \mathcal{D}_B \Psi = 0, \quad (2.45)$$

$$\gamma^A \mathcal{D}_A \Psi = 0, \quad (2.46)$$

$$[\Psi^\alpha, \Psi^\beta]_\star (\mathbf{C} \gamma^{(5)} \gamma^p)_{\alpha\beta} = 0, \quad (2.47)$$

$$\partial_\Psi \text{Tr} [V_\star(\Psi)] = 0. \quad (2.48)$$

Especially, when $g_{\text{HS}}^{-2} \text{Tr}[V_\star(\Psi)]$ vanishes (either in the large g_{HS} limit or at the bottom of the potential), (2.42) and (2.43) become precisely the equations of motion of the ‘pure’ DFT Lagrangian (2.30), and hence any solution to the above BPS equations is qualified as a ‘vacuum’ configuration.

The ‘algebraic’ condition (2.47) is equivalent, from (2.6), to

$$[\Psi^\alpha, \Psi^\beta]_\star = \frac{1}{4}[(1 + \gamma^{(5)})\mathbf{C}^{-1}]^{\alpha\beta} \mathcal{Q}_+ + \frac{1}{4}[(1 - \gamma^{(5)})\mathbf{C}^{-1}]^{\alpha\beta} \mathcal{Q}_-, \quad (2.49)$$

where we set

$$\begin{aligned} \mathcal{Q}_+ &:= -\bar{\Psi} \star (1 + \gamma^{(5)}) \star \Psi = -\frac{1}{2}[\Psi^\alpha, \Psi^\beta]_\star [\mathbf{C}(1 + \gamma^{(5)})]_{\alpha\beta}, \\ \mathcal{Q}_- &:= -\bar{\Psi} \star (1 - \gamma^{(5)}) \star \Psi = -\frac{1}{2}[\Psi^\alpha, \Psi^\beta]_\star [\mathbf{C}(1 - \gamma^{(5)})]_{\alpha\beta} = (\mathcal{Q}_+)^\dagger. \end{aligned} \quad (2.50)$$

- The HS-DFT BPS equations, (2.42)–(2.48), and consequently the full set of the equations of motion, (2.36)–(2.41), admit the following *vacuum solution*, characterized by linear DFT-dilaton and constant, *i.e.* flat, DFT-vielbein (*c.f.* section 3.7),

$$\mathring{d} = \mathring{N}_A x^A, \quad \partial_A \mathring{V}_{Bp} = 0, \quad \mathring{\mathcal{W}}_A = -\frac{1}{3} \bar{\zeta} \gamma_{AB} \zeta \mathring{N}^B, \quad \mathring{\Psi}^\alpha = m^{\frac{3}{2}} \mathbf{Re}(\zeta^\alpha), \quad (2.51)$$

provided the potential assumes its minimum or $g_{\text{HS}}^{-2} \text{Tr}[V_\star(\mathring{\Psi})] = 0$. Here $\mathring{N}_A = \partial_A \mathring{d}$ is a constant $\mathbf{O}(4, 4)$ null vector, $\mathring{N}_A \mathring{N}^A = 0$, normalized to meet

$$\Lambda_{\text{DFT}} = -4 \mathring{N}_p \mathring{N}^p. \quad (2.52)$$

Further, m is a constant mass parameter introduced to match the $\frac{3}{2}$ mass dimension of Ψ , and $\mathbf{Re}(\zeta^\alpha)$ denotes the ‘real’ part of ζ^α ,

$$\mathbf{Re}(\zeta^\alpha) = \frac{1}{2}(\zeta^\alpha + \bar{\zeta}_\beta \mathbf{C}^{-1\beta\alpha}). \quad (2.53)$$

In particular, the vacuum solution gives

$$\mathring{\mathcal{Q}}_+ = \mathring{\mathcal{Q}}_- = m^3. \quad (2.54)$$

We emphasize that, in our proposal of HS-DFT, there is no restriction on Λ_{DFT} : any value with any sign is allowed, as is the case with half-maximal supersymmetric gauged double field theories [52]. Accordingly, the above linear DFT-dilaton vacuum calls for a space-like, null-like or time-like four-dimensional vector, \mathring{N}^p , in each case of $\Lambda_{\text{DFT}} < 0$, $\Lambda_{\text{DFT}} = 0$ or $\Lambda_{\text{DFT}} > 0$ respectively.

- On the other hand, imposing extra conditions: (i) the sectioning condition on the HS gauge potential,

$$\mathcal{W}^A \partial_A \equiv 0, \quad \mathcal{W}^A \mathcal{W}_A \equiv 0, \quad (2.55)$$

(ii) the deformed oscillator relations on the bosonic Majorana spinor, as solutions to (2.48),

$$\{(1+\gamma^{(5)})\Psi, \mathcal{Q}_+\}_\star \equiv 2m^3(1+\gamma^{(5)})\Psi, \quad \{(1-\gamma^{(5)})\Psi, \mathcal{Q}_-\}_\star \equiv 2m^3(1-\gamma^{(5)})\Psi, \quad (2.56)$$

and (iii) a twisted reality condition on \mathcal{Q}_\pm in terms of the inner Klein operators defined in (3.30),

$$\mathcal{Q}_+ - m^3 = (\mathcal{Q}_- - m^3) \star \mathbf{k} \star \bar{\mathbf{k}}, \quad (2.57)$$

we may reduce the HS-DFT BPS conditions, (2.42)–(2.48), to the bosonic Vasiliev equations in four dimensions, *c.f.* (3.156)–(3.159), (3.180), (3.183). The above linear dilaton vacuum (2.51) does not meet (2.55). Moreover, unlike the Vasiliev theory, its HS gauge potential, \mathcal{W}_A , does not contain a gravitational spin-two ‘vielbein’ (only the spin connection appears, *c.f.* (3.128)). Hence, it corresponds to a genuine HS-DFT background, particularly realizing the open string interpretation [17].

- In particular, the following two choices of the potential are of interest,

$$V_\star^{\text{YM}}(\Psi) = \frac{1}{2} \lambda_{\text{YM}} [(\mathcal{Q}_+ - m^3) \star (\mathcal{Q}_+ - m^3) + (\mathcal{Q}_- - m^3) \star (\mathcal{Q}_- - m^3)], \quad (2.58)$$

and

$$V_\star^{\text{def. osc.}}(\Psi) = \frac{1}{2} \lambda_{\text{def. osc.}} (\mathcal{R}_+ \star \mathcal{R}_+ + \mathcal{R}_- \star \mathcal{R}_-), \quad (2.59)$$

where for the latter we set, similarly to (2.50),

$$\mathcal{R}_\pm := -\bar{\Upsilon}_\pm \star (1 \pm \gamma^{(5)}) \star \Upsilon_\pm, \quad \Upsilon_\pm := \{(1 \pm \gamma^{(5)})\Psi, (\mathcal{Q}_\pm - m^3)\}_\star. \quad (2.60)$$

We call the former “Yang-Mills” potential and the latter “deformed oscillator” potential, as the former is essentially Ψ -commutator squared, *c.f.* (2.50), up to surface integral over the internal space and constant shift, and the latter is designed to make the deformed oscillator relations (2.56), *i.e.* $\Upsilon_+ = \Upsilon_- = 0$, to solve the algebraic BPS condition (2.48), *i.e.* $\partial_\Psi \text{Tr}[V_\star^{\text{def. osc.}}(\Psi)] = 0$.

In the case of the “Yang-Mills” potential, we have $\mathcal{Q}_\pm \rightarrow m^3$ in the low energy limit, and hence the deformed oscillator relations can be approximately achieved, while it may realize a Brout-Englert-Higgs mechanism. It is worth while to note for the last algebraic BPS condition (2.48),

$$\partial_\Psi \text{Tr}[V_\star^{\text{YM}}(\Psi)] = 0 \quad \Longleftrightarrow \quad [(1+\gamma^{(5)})\Psi, \mathcal{Q}_+] = 0, \quad \& \quad [(1-\gamma^{(5)})\Psi, \mathcal{Q}_-] = 0. \quad (2.61)$$

In summary, the HS-DFT extension of the bosonic Vasiliev equations consists of (2.44), (2.45), (2.46), (2.47), (2.55) and (2.56); while the BPS conditions (2.44)–(2.48), together with the “YM” potential (2.58), may lead to a possible open string realization of higher spin theory.

3 Exposition

Here we provide some complementary explanations of the HS-DFT proposed in the previous section.

3.1 Doubled-yet-gauged coordinates and DFT-diffeomorphisms

The section condition (2.7) decomposes into the linear weak constraint and the quadratic strong constraint:

$$\partial_A \partial^A \phi = 0, \quad \partial_A \phi \partial^A \varphi = 0. \quad (3.1)$$

Here and in this subsection, ϕ and φ are arbitrary functions and their derivatives in the HS-DFT we construct. Demanding the weak constraint to hold also for a product, *i.e.* $\partial_A \partial^A (\phi \varphi) = 0$, we are led to the strong constraint. This explains the nomenclature behind the terms, ‘weak’ and ‘strong’. On the other hand, if we substitute $\partial^B \phi$ and $\partial_C \phi$ into ϕ and φ , the strong constraint actually implies the weak constraint, since as an 8×8 matrix, $\partial_A \partial^B \phi$ is nilpotent and hence is traceless [49]. Furthermore, as can be shown easily from the power series expansion [48], the strong constraint means that all the functions in the theory are invariant under the following ‘shifts’ of the doubled coordinates,

$$\Phi(x + \Delta) = \Phi(x), \quad \Delta^A = \phi \partial^A \varphi. \quad (3.2)$$

This simple observation reveals the geometric meaning behind the section condition that, *the doubled coordinates are in fact gauged* through an equivalence relation [48, 49],

$$x^A \sim x^A + \phi \partial^A \varphi. \quad (3.3)$$

It is then not a point in the doubled coordinate system but each equivalence class or a gauge orbit that represents a single physical point in the undoubled spacetime (*c.f.* [53–56] for further discussions).⁷

With the decomposition of the doubled coordinates, $x^A = (\tilde{x}_\mu, x^\nu)$, with respect to the $\mathbf{O}(4, 4)$ invariant metric, \mathcal{J}_{AB} (2.1), and from $\partial_A \partial^A = 2 \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu}$, the section condition can be conveniently solved by requiring that all the fields are independent of the “dual” coordinates,

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0. \quad (3.4)$$

The general solutions are then given by the $\mathbf{O}(4, 4)$ rotations of this specific solution. However, unless mentioned explicitly, we shall not assume any particular solution to the section condition such as (3.4), as

⁷This idea of ‘Coordinate Gauge Symmetry’ can be realized on a string worldsheet as a conventional gauge symmetry of a doubled sigma model, by introducing a corresponding gauge potential [49]. Integrating out the auxiliary gauge potential, the doubled sigma model reduces to the standard undoubled string action on an arbitrarily curved NS-NS background.

we intend to keep the manifest $\mathbf{O}(4, 4)$ covariance throughout the proposal.

The diffeomorphisms on the doubled-yet-gauged coordinate system, *i.e.* *DFT-diffeomorphisms*, are generated by the generalized Lie derivative [19, 51]. Acting on an arbitrary covariant tensor with weight ω_T , it reads

$$\hat{\mathcal{L}}_X T_{A_1 \dots A_n} := X^B \partial_B T_{A_1 \dots A_n} + \omega_T \partial_B X^B T_{A_1 \dots A_n} + \sum_{i=1}^n (\partial_{A_i} X_B - \partial_B X_{A_i}) T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}, \quad (3.5)$$

where other types of indices are suppressed for simplicity. Specifically, for the field content of HS-DFT (2.10), the DFT-diffeomorphisms are given by

$$\begin{aligned} \delta_X d &= -\frac{1}{2} e^{2d} \hat{\mathcal{L}}_X (e^{-2d}) = X^A \partial_A d - \frac{1}{2} \partial_A X^A, \\ \delta_X V_{Ap} &= \hat{\mathcal{L}}_X V_{Ap} = X^B \partial_B V_{Ap} + (\partial_A X^B - \partial^B X_A) V_{Bp}, \\ \delta_X \mathcal{W}_A &= \hat{\mathcal{L}}_X \mathcal{W}_A = X^B \partial_B \mathcal{W}_A + (\partial_A X^B - \partial^B X_A) \mathcal{W}_B, \\ \delta_X \Psi^\alpha &= \hat{\mathcal{L}}_X \Psi^\alpha = X^A \partial_A \Psi^\alpha. \end{aligned} \quad (3.6)$$

It is worth while to note that the $\mathbf{O}(4, 4)$ invariant metric is compatible with the generalized Lie derivative, $\hat{\mathcal{L}}_X \mathcal{J}_{AB} = 0$, and the commutator of the generalized Lie derivatives is closed, up to the section condition, by so-called the C-bracket [19, 21],

$$[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] = \hat{\mathcal{L}}_{[X, Y]_C}, \quad [X, Y]_C^A = X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B. \quad (3.7)$$

3.2 Spin(1, 3) Clifford algebra: Majorana and Dirac spinors

Combining \mathbf{A} and \mathbf{C} , (2.3), (2.4), we obtain

$$\mathbf{B}_{\alpha\beta} = \mathbf{C}_{\alpha\gamma} \mathbf{A}^\gamma{}_\beta, \quad (\gamma^p)^* = +\mathbf{B} \gamma^p \mathbf{B}^{-1}, \quad (\gamma^{(5)})^* = -\mathbf{B} \gamma^{(5)} \mathbf{B}^{-1}. \quad (3.8)$$

Consequently, with $\mathbf{A} = \mathbf{A}^\dagger = \mathbf{C}^{-1} \mathbf{B}$, we get

$$(\mathbf{C}^{-1})^\dagger = +\mathbf{B} \mathbf{C}^{-1} \mathbf{B}, \quad (\gamma^{(5)} \mathbf{C}^{-1})^\dagger = -\mathbf{B} \gamma^{(5)} \mathbf{C}^{-1} \mathbf{B}. \quad (3.9)$$

It is crucial to note that \mathbf{B} is unitary and symmetric, satisfying [57],

$$\mathbf{B}^* \mathbf{B} = 1, \quad \mathbf{B}_{\alpha\beta} = \mathbf{B}_{\beta\alpha}. \quad (3.10)$$

This property enables us to impose the Majorana conditions on the spinor, Ψ^α ,

$$\bar{\Psi} = \Psi^\dagger \mathbf{A} = \Psi^T \mathbf{C} \quad \Longleftrightarrow \quad \Psi^* = \mathbf{B} \Psi, \quad (3.11)$$

in a self-consistent manner, as

$$\Psi = (\Psi^*)^* = (\mathbf{B} \Psi)^* = \mathbf{B}^* \mathbf{B} \Psi. \quad (3.12)$$

Further, we can decompose the complex Dirac spinor, ζ^α , which sets the non-commutative internal space (2.8), into the ‘real’ and ‘imaginary’ parts,

$$\zeta = \mathbf{Re}(\zeta) + \mathbf{Im}(\zeta), \quad (3.13)$$

where

$$\begin{aligned} \mathbf{Re}(\zeta) &= \frac{1}{2} [\zeta + (\mathbf{B}\zeta)^*] \equiv \zeta_+ & \Longleftrightarrow & \zeta_+^\alpha = \frac{1}{2} (\zeta^\alpha + \bar{\zeta}_\beta C^{-1\beta\alpha}), \\ \mathbf{Im}(\zeta) &= \frac{1}{2} [\zeta - (\mathbf{B}\zeta)^*] \equiv \zeta_- & \Longleftrightarrow & \zeta_-^\alpha = \frac{1}{2} (\zeta^\alpha - \bar{\zeta}_\beta C^{-1\beta\alpha}). \end{aligned} \quad (3.14)$$

These are ‘real’ (Majorana) and ‘imaginary’ (pseudo-Majorana) in the following sense,

$$(\mathbf{B}\zeta_+)^* = +\zeta_+, \quad (\mathbf{B}\zeta_-)^* = -\zeta_-, \quad \bar{\zeta}_\pm = \zeta_\pm^\dagger \mathbf{A} = \pm \zeta_\pm^T \mathbf{C}. \quad (3.15)$$

For the bosonic Dirac spinors, $\zeta, \bar{\zeta}$, we have then⁸

$$\bar{\zeta} \zeta = (\bar{\zeta} \zeta)^\dagger = (\bar{\zeta} \zeta)^* = \bar{\zeta}_+ \zeta_- + \bar{\zeta}_- \zeta_+ = 2\zeta_+^T \mathbf{C} \zeta_- = -2\zeta_-^T \mathbf{C} \zeta_+. \quad (3.16)$$

Our spacetime signature, $(-+++)$, admits the real, *i.e.* Majorana, representation of the gamma matrices, which means that, if desired, we may put $\mathbf{B} \equiv 1$, *c.f.* (3.168).

Both the higher spin gauge potential, \mathcal{W}_A , and the bosonic Majorana spinor, Ψ^α , are HS algebra valued, such that they depend on all the coordinates, $x^A, \zeta^\alpha, \bar{\zeta}_\beta$ generically, and can be expanded by the internal spinorial coordinates,⁹

$$\begin{aligned} \mathcal{W}_A(x, \zeta, \bar{\zeta}) &= \sum_{m,n} \frac{1}{m!n!} \zeta^{\alpha_1} \zeta^{\alpha_2} \dots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \dots \bar{\zeta}_{\beta_n} \mathcal{W}_{A\alpha_1\alpha_2\dots\alpha_m}{}^{\beta_1\beta_2\dots\beta_n}(x), \\ \Psi^\alpha(x, \zeta, \bar{\zeta}) &= \sum_{m,n} \frac{1}{m!n!} \zeta^{\alpha_1} \zeta^{\alpha_2} \dots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \dots \bar{\zeta}_{\beta_n} \Psi^\alpha{}_{\alpha_1\alpha_2\dots\alpha_m}{}^{\beta_1\beta_2\dots\beta_n}(x). \end{aligned} \quad (3.17)$$

⁸On the other hand, for a *fermionic* (*i.e.* odd Grassmannian) Dirac spinor, say ψ , the relation (3.16) gets modified,

$$\bar{\psi} \psi = \bar{\psi}_+ \psi_+ + \bar{\psi}_- \psi_- = \psi_+^T \mathbf{C} \psi_+ - \psi_-^T \mathbf{C} \psi_-.$$

⁹For a duality manifest alternative approach to higher spin fields through twistor variables, see Ref. [58] by Cederwall.

Naturally, the component fields satisfy symmetric properties,

$$\begin{aligned}\mathcal{W}_{A\alpha_1\alpha_2\cdots\alpha_m}{}^{\beta_1\beta_2\cdots\beta_n}(x) &= \mathcal{W}_{A(\alpha_1\alpha_2\cdots\alpha_m)}{}^{(\beta_1\beta_2\cdots\beta_n)}(x), \\ \Psi_{\alpha_1\alpha_2\cdots\alpha_m}{}^{\beta_1\beta_2\cdots\beta_n}(x) &= \Psi_{(\alpha_1\alpha_2\cdots\alpha_m)}{}^{(\beta_1\beta_2\cdots\beta_n)}(x),\end{aligned}\tag{3.18}$$

and also, from the anti-Hermiticity of \mathcal{W}_A and the Majorana property of Ψ^α , a set of reality conditions follows

$$\begin{aligned}\mathbf{A}^{\beta_1}_{\delta_1} \cdots \mathbf{A}^{\beta_n}_{\delta_n} \mathcal{W}_{A\alpha_1\cdots\alpha_m}{}^{\delta_1\cdots\delta_n} &= -(\mathbf{A}^{\alpha_1}_{\gamma_1} \cdots \mathbf{A}^{\alpha_m}_{\gamma_m} \mathcal{W}_{A\beta_1\cdots\beta_n}{}^{\gamma_1\cdots\gamma_m})^*, \\ \mathbf{B}_{\alpha\beta} \mathbf{A}^{\beta_1}_{\delta_1} \cdots \mathbf{A}^{\beta_n}_{\delta_n} \Psi_{\alpha_1\cdots\alpha_m}{}^{\delta_1\cdots\delta_n} &= +(\mathbf{A}^{\alpha_1}_{\gamma_1} \cdots \mathbf{A}^{\alpha_m}_{\gamma_m} \Psi_{\beta_1\cdots\beta_n}{}^{\gamma_1\cdots\gamma_m})^*.\end{aligned}\tag{3.19}$$

These reality conditions relate the component fields pairwise, *i.e.* $m \leftrightarrow n$.

Similarly, the higher spin gauge parameter (2.15) can be expanded,

$$\begin{aligned}\mathcal{T}(x, \zeta, \bar{\zeta}) &= \sum_{m,n} \frac{1}{m!n!} \zeta^{\alpha_1} \zeta^{\alpha_2} \cdots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \cdots \bar{\zeta}_{\beta_n} \mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_m}{}^{\beta_1\beta_2\cdots\beta_n}(x), \\ \mathbf{A}^{\beta_1}_{\delta_1} \cdots \mathbf{A}^{\beta_n}_{\delta_n} \mathcal{T}_{\alpha_1\cdots\alpha_m}{}^{\delta_1\cdots\delta_n} &= -(\mathbf{A}^{\alpha_1}_{\gamma_1} \cdots \mathbf{A}^{\alpha_m}_{\gamma_m} \mathcal{T}_{\beta_1\cdots\beta_n}{}^{\gamma_1\cdots\gamma_m})^*.\end{aligned}\tag{3.20}$$

And clearly, the component fields of different ranks of (m, n) (3.18) may transform to each other under the higher spin gauge symmetry, (2.15).

3.3 Wick ordered star product

We recall the definition of the star product (2.9),

$$f(x, \zeta, \bar{\zeta}) \star g(x, \zeta, \bar{\zeta}) = f(x, \zeta, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial\zeta^\alpha} \frac{\overrightarrow{\partial}}{\partial\bar{\zeta}_\alpha}\right) g(x, \zeta, \bar{\zeta}).\tag{3.21}$$

From the Hermiticity of the matrix, \mathbf{A} , we note

$$(f \star g)^\dagger = g^\dagger \star f^\dagger.\tag{3.22}$$

The star product can be equivalently reformulated as integrals,

$$f(x, \zeta, \bar{\zeta}) \star g(x, \zeta, \bar{\zeta}) = \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ e^{\bar{\lambda}_+\rho_+} f(x, \zeta + \lambda_+, \bar{\zeta}) g(x, \zeta, \bar{\zeta} + \bar{\rho}_+),\tag{3.23}$$

where λ_+ and ρ_+ are two separate bosonic Majorana spinors to integrate,

$$\bar{\lambda}_+ = \lambda_+^\dagger \mathbf{A} = \lambda_+^T \mathbf{C}, \quad \bar{\rho}_+ = \rho_+^\dagger \mathbf{A} = \rho_+^T \mathbf{C}. \quad (3.24)$$

In particular, the product, $\bar{\lambda}_+ \rho_+$, is pure imaginary,

$$(\bar{\lambda}_+ \rho_+)^\dagger = \bar{\rho}_+ \lambda_+ = -\bar{\lambda}_+ \rho_+, \quad (3.25)$$

and its exponentiation can serve as an integrand for an integral representation of the Dirac delta function,

$$\frac{1}{(2\pi)^4} \int d^4 \rho_+ e^{\bar{\lambda}_+ \rho_+} = \delta(\lambda_+). \quad (3.26)$$

The equivalence of the two expressions for the star product, (3.21) and (3.23), can be then straightforwardly established.¹⁰ We present our own proof in the Appendix (A.1).

The star product satisfies the associativity,

$$f \star (g \star h) = (f \star g) \star h = f \star g \star h, \quad (3.27)$$

which can be also shown directly from the integral expression of the star product, *c.f.* Appendix (A.2).

Further, the star product over the $(\zeta^\alpha, \bar{\zeta}_\beta)$ internal space is isomorphic to the Wick ordered operator formalism,

$$: f(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : = \hat{\mathcal{O}}[f(\zeta, \bar{\zeta}) \star g(\zeta, \bar{\zeta})], \quad (3.28)$$

where any hatted object is an operator; the colon denotes the Wick ordering to place all the unbarred (annihilation) operators, $\hat{\zeta}^\alpha$, to the right and the barred (creation) operators, $\hat{\bar{\zeta}}_\beta$, to the left; and for an arbitrary function of the internal commuting coordinates, $f(\zeta, \bar{\zeta})$, the corresponding operator, $\hat{\mathcal{O}}[f(\zeta, \bar{\zeta})]$, is defined subject to the Wick ordering prescription,

$$\hat{\mathcal{O}}[f(\zeta, \bar{\zeta})] = : f(\hat{\zeta}, \hat{\bar{\zeta}}) :. \quad (3.29)$$

We refer readers to Appendix (A.3) for our own proof of the isomorphism.

We define a pair of inner Klein operators, \mathbf{k} and $\bar{\mathbf{k}}$, exponentiating the quadratic forms of the bosonic internal spinors [3]:

$$\mathbf{k} := e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta}, \quad \bar{\mathbf{k}} := e^{-\bar{\zeta}(1-\gamma^{(5)})\zeta} = \mathbf{k}^\dagger. \quad (3.30)$$

¹⁰Yet, the integral formula (3.23) may have better convergence property than the differential one (3.21).

For an arbitrary function, $f(x, \zeta, \bar{\zeta})$, we compute to acquire

$$\begin{aligned}
& \mathbf{k} \star f(x, \zeta, \bar{\zeta}) \star \mathbf{k} \\
&= \{ \mathbf{k} \star f(x, \zeta, \bar{\zeta}) \} \star \mathbf{k} \\
&= \left\{ e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \exp\left(\frac{\overleftarrow{\partial}}{\partial\zeta^\alpha} \frac{\overrightarrow{\partial}}{\partial\zeta_\alpha}\right) f(x, \zeta, \bar{\zeta}) \right\} \exp\left(\frac{\overleftarrow{\partial}}{\partial\zeta^\beta} \frac{\overrightarrow{\partial}}{\partial\zeta_\beta}\right) e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \\
&= \left\{ e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \exp\left(-[\bar{\zeta}(1+\gamma^{(5)})]_\alpha \frac{\overrightarrow{\partial}}{\partial\zeta_\alpha}\right) f(x, \zeta, \bar{\zeta}) \right\} \exp\left(-\frac{\overleftarrow{\partial}}{\partial\zeta^\beta} [(1+\gamma^{(5)})\zeta]^\beta\right) e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \\
&= \left\{ e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} f(x, \zeta, -\bar{\zeta}\gamma^{(5)}) \right\} \exp\left(-\frac{\overleftarrow{\partial}}{\partial\zeta^\beta} [(1+\gamma^{(5)})\zeta]^\beta\right) e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \\
&= \left\{ e^{\bar{\zeta}(\gamma^{(5)}+1)\zeta} f(x, -\gamma^{(5)}\zeta, -\bar{\zeta}\gamma^{(5)}) \right\} e^{-\bar{\zeta}(1+\gamma^{(5)})\zeta} \\
&= f(x, -\gamma^{(5)}\zeta, -\bar{\zeta}\gamma^{(5)}) .
\end{aligned} \tag{3.31}$$

Similarly, by replacing $\gamma^{(5)}$ by $-\gamma^{(5)}$, we get

$$\bar{\mathbf{k}} \star f(x, \zeta, \bar{\zeta}) \star \bar{\mathbf{k}} = f(x, \gamma^{(5)}\zeta, \bar{\zeta}\gamma^{(5)}) . \tag{3.32}$$

Then, considering various cases of $f(x, \zeta, \bar{\zeta})$, such as constant, or the Klein operators themselves, we can obtain the crucial properties of the Klein operators:

$$\mathbf{k} \star \mathbf{k} = 1, \quad \bar{\mathbf{k}} \star \bar{\mathbf{k}} = 1, \quad \mathbf{k} \star \bar{\mathbf{k}} = \bar{\mathbf{k}} \star \mathbf{k}, \tag{3.33}$$

and

$$\mathbf{k} \star f(x, \zeta, \bar{\zeta}) = f(x, -\gamma^{(5)}\zeta, -\bar{\zeta}\gamma^{(5)}) \star \mathbf{k}, \quad \bar{\mathbf{k}} \star f(x, \zeta, \bar{\zeta}) = f(x, +\gamma^{(5)}\zeta, +\bar{\zeta}\gamma^{(5)}) \star \bar{\mathbf{k}}. \tag{3.34}$$

Further, combining the two Klein operators, we get

$$\mathbf{k} \star \bar{\mathbf{k}} \star f(x, \zeta, \bar{\zeta}) = f(x, -\zeta, -\bar{\zeta}) \star \mathbf{k} \star \bar{\mathbf{k}}. \tag{3.35}$$

Explicitly, in a similar fashion to (3.31), we have

$$\mathbf{k} \star \bar{\mathbf{k}} = \bar{\mathbf{k}} \star \mathbf{k} = e^{-2\bar{\zeta}\zeta}. \tag{3.36}$$

The ‘bosonic’ truncation of the HS-DFT is then achieved by requiring

$$\mathbf{k} \star \bar{\mathbf{k}} \star \mathcal{W}_A - \mathcal{W}_A \star \mathbf{k} \star \bar{\mathbf{k}} = 0, \quad \mathbf{k} \star \bar{\mathbf{k}} \star \Psi^\alpha + \Psi^\alpha \star \mathbf{k} \star \bar{\mathbf{k}} = 0, \quad \mathbf{k} \star \bar{\mathbf{k}} \star \mathcal{T} - \mathcal{T} \star \mathbf{k} \star \bar{\mathbf{k}} = 0, \quad (3.37)$$

and hence, equivalently,

$$\mathcal{W}_A(x, -\zeta, -\bar{\zeta}) = +\mathcal{W}_A(x, \zeta, \bar{\zeta}), \quad \Psi^\alpha(x, -\zeta, -\bar{\zeta}) = -\Psi^\alpha(x, \zeta, \bar{\zeta}), \quad \mathcal{T}(x, -\zeta, -\bar{\zeta}) = +\mathcal{T}(x, \zeta, \bar{\zeta}). \quad (3.38)$$

Basically the truncation implies that the HS gauge potential, \mathcal{W}_A , and the HS gauge parameter, \mathcal{T} , are restricted to be even functions of $\zeta, \bar{\zeta}$, while the Majorana spinor, Ψ^α , should be an odd function.

It is straightforward to verify

$$\begin{aligned} [\bar{\zeta} \gamma^r \zeta, \zeta^\alpha]_\star &= -(\gamma^r \zeta)^\alpha, & [\bar{\zeta} \gamma^r \zeta, \bar{\zeta}_\alpha]_\star &= (\bar{\zeta} \gamma^r)_\alpha, \\ [\bar{\zeta} \gamma^{pq} \zeta, \zeta^\alpha]_\star &= -(\gamma^{pq} \zeta)^\alpha, & [\bar{\zeta} \gamma^{pq} \zeta, \bar{\zeta}_\alpha]_\star &= (\bar{\zeta} \gamma^{pq})_\alpha, \end{aligned} \quad (3.39)$$

and

$$\begin{aligned} [\bar{\zeta} \gamma^r \zeta, \bar{\zeta} \gamma^s \zeta]_\star &= \bar{\zeta} (\gamma^r \gamma^s - \gamma^s \gamma^r) \zeta = 2 \bar{\zeta} \gamma^{rs} \zeta, \\ [\bar{\zeta} \gamma^{pq} \zeta, \bar{\zeta} \gamma^r \zeta]_\star &= \bar{\zeta} (\gamma^{pq} \gamma^r - \gamma^r \gamma^{pq}) \zeta = 2 (\eta^{qr} \bar{\zeta} \gamma^p \zeta - \eta^{pr} \bar{\zeta} \gamma^q \zeta), \\ [\bar{\zeta} \gamma^{pq} \zeta, \bar{\zeta} \gamma^{rs} \zeta]_\star &= \bar{\zeta} (\gamma^{pq} \gamma^{rs} - \gamma^{rs} \gamma^{pq}) \zeta = 2 (\eta^{qr} \bar{\zeta} \gamma^{ps} \zeta - \eta^{pr} \bar{\zeta} \gamma^{qs} \zeta + \eta^{ps} \bar{\zeta} \gamma^{qr} \zeta - \eta^{qs} \bar{\zeta} \gamma^{pr} \zeta). \end{aligned} \quad (3.40)$$

Therefore, if we define two sets of Wick ordered quadratic operators,

$$\hat{P}^r := \frac{1}{2} \hat{\bar{\zeta}} \gamma^r \hat{\zeta} = -(\hat{P}^r)^\dagger, \quad \hat{M}^{pq} := \frac{1}{2} \hat{\bar{\zeta}} \gamma^{pq} \hat{\zeta} = -(\hat{M}^{pq})^\dagger, \quad (3.41)$$

Eq.(3.39) gives

$$\begin{aligned} [\hat{P}^r, \hat{\zeta}^\alpha] &= -\frac{1}{2} (\gamma^r \hat{\zeta})^\alpha, & [\hat{P}^r, \hat{\bar{\zeta}}_\alpha] &= \frac{1}{2} (\bar{\zeta} \gamma^r)_\alpha, \\ [\hat{M}^{pq}, \hat{\zeta}^\alpha] &= -\frac{1}{2} (\gamma^{pq} \hat{\zeta})^\alpha, & [\hat{M}^{pq}, \hat{\bar{\zeta}}_\alpha] &= \frac{1}{2} (\bar{\zeta} \gamma^{pq})_\alpha, \end{aligned} \quad (3.42)$$

and Eq.(3.40) realizes an $\mathfrak{so}(2, 3)$ algebra,

$$\begin{aligned} [\hat{P}^r, \hat{P}^s] &= \hat{M}^{rs}, \\ [\hat{M}^{pq}, \hat{P}^r] &= \eta^{qr} \hat{P}^p - \eta^{pr} \hat{P}^q, \\ [\hat{M}^{pq}, \hat{M}^{rs}] &= \eta^{qr} \hat{M}^{ps} - \eta^{pr} \hat{M}^{qs} + \eta^{ps} \hat{M}^{qr} - \eta^{qs} \hat{M}^{pr}. \end{aligned} \quad (3.43)$$

Finally, it is worth while to note the star commutator relations for the real (Majorana) and the imaginary (pseudo-Majorana) spinors,

$$[\zeta_+^\alpha, \zeta_+^\beta]_\star = +\frac{1}{2}\mathbf{C}^{-1\alpha\beta}, \quad [\zeta_-^\alpha, \zeta_-^\beta]_\star = -\frac{1}{2}\mathbf{C}^{-1\alpha\beta}, \quad [\zeta_+^\alpha, \zeta_-^\beta]_\star = 0, \quad (3.44)$$

which are equivalent to (2.8).

3.4 DFT-vielbein, projectors and the master derivative

Here we review the semi-covariant differential geometry developed for double field theory [28], with the intention of incorporating the higher spin gauge symmetry. Firstly, we recall the defining property of the DFT-vielbein (2.11),

$$V_{Ap}V^A{}_q = \eta_{pq}. \quad (3.45)$$

If we view V_{Ap} as an 8×4 matrix and *assume* that its upper 4×4 block is non-degenerate, the defining condition (3.45) can be generically solved by the following parametrization,

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^\mu \\ (B + e)_{\nu p} \end{pmatrix}, \quad (3.46)$$

where, with respect to the aforementioned particular choice of the section, $\frac{\partial}{\partial \bar{x}_\mu} \equiv 0$ (3.4), $e_\mu{}^p$ corresponds to an ordinary vierbein and thus sets a Riemannian metric,

$$g_{\mu\nu} = e_\mu{}^p e_\nu{}^q \eta_{pq}. \quad (3.47)$$

Further, in (3.46), we put $B_{\nu p} = B_{\nu\sigma}(e^{-1})_p{}^\sigma$ with a skew-symmetric two-form field, $B_{\mu\nu} = -B_{\nu\mu}$. On the other hand, if the upper block is degenerate, the DFT-vielbein cannot be parametrized as above. It should be solved in a different manner, and it generically leads to a ‘non-Riemannian’ stringy background

which does not admit any Riemannian interpretation [38, 49].¹¹ Unless explicitly stated, hereafter we shall not assume any particular parametrization of the DFT-vielbein like (3.46). Only the defining property (3.45) needs to be assumed and suffices.

The DFT-vielbein produces a pair of projectors. Firstly, we set

$$P_{AB} := V_{Ap}V_B^p, \quad (3.48)$$

which is, by construction with (3.45), a symmetric projector,

$$P_{AB} = P_{BA}, \quad P_A{}^B P_B{}^C = P_A{}^C. \quad (3.49)$$

The complementary projector is subsequently defined,

$$\bar{P}_A{}^B := \delta_A{}^B - P_A{}^B, \quad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C, \quad (3.50)$$

such that both P_{AB} and \bar{P}_{AB} are symmetric projectors, being orthogonal and complete,

$$P_A{}^B \bar{P}_B{}^C = 0, \quad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}. \quad (3.51)$$

Unlike the supersymmetric double field theories [35, 36], it is unnecessary to introduce a separate $\mathbf{Spin}(3, 1)$ DFT-vielbein, $\bar{V}_{A\bar{p}}$, and to define the orthogonal projector as its “square”, *i.e.* $\bar{P}_{AB} = \bar{V}_A{}^{\bar{p}}\bar{V}_{B\bar{p}}$, which would be analogous to (3.48). In the current proposal of the HS-DFT, we make use of only one spin group, such that Ψ^α is not $\mathbf{Spin}(3, 1)$ but $\mathbf{Spin}(1, 3)$ Majorana spinor.

The two-index projectors further generate ‘six-index’ projectors [28]: with $D = 4$,

$$\begin{aligned} \mathcal{P}_{ABC}{}^{DEF} &:= P_A{}^D P_{[B}{}^{[E} P_{C]}{}^{F]} + \frac{2}{D-1} P_{A[B} P_{C]}{}^{[E} P^{F]D}, & \mathcal{P}_{ABC}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} &= \mathcal{P}_{ABC}{}^{GHI}, \\ \bar{\mathcal{P}}_{ABC}{}^{DEF} &:= \bar{P}_A{}^D \bar{P}_{[B}{}^{[E} \bar{P}_{C]}{}^{F]} + \frac{2}{D-1} \bar{P}_{A[B} \bar{P}_{C]}{}^{[E} \bar{P}^{F]D}, & \bar{\mathcal{P}}_{ABC}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} &= \bar{\mathcal{P}}_{ABC}{}^{GHI}, \end{aligned} \quad (3.52)$$

which are symmetric and traceless in the following sense,

$$\begin{aligned} \mathcal{P}_{ABCDEF} &= \mathcal{P}_{DEFABC}, & \mathcal{P}_{ABCDEF} &= \mathcal{P}_{A[BC]D[EF]}, & P^{AB} \mathcal{P}_{ABCDEF} &= 0, \\ \bar{\mathcal{P}}_{ABCDEF} &= \bar{\mathcal{P}}_{DEFABC}, & \bar{\mathcal{P}}_{ABCDEF} &= \bar{\mathcal{P}}_{A[BC]D[EF]}, & \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} &= 0. \end{aligned} \quad (3.53)$$

¹¹See also [50] for U-duality manifest generalized metrics.

We are now ready to explain the properties of the *master derivative* (2.22),

$$\mathcal{D}_A := \partial_A + \Gamma_A(x) + \Phi_A(x) + \mathcal{W}_A(x, \zeta, \bar{\zeta}), \quad (3.54)$$

which includes *the semi-covariant derivative* introduced in [27, 28] for the DFT-diffeomorphisms,

$$\nabla_A := \partial_A + \Gamma_A. \quad (3.55)$$

Explicitly, acting on a generic covariant tensor (3.5), the semi-covariant derivative reads

$$\nabla_C T_{A_1 A_2 \dots A_n} = \partial_C T_{A_1 A_2 \dots A_n} - \omega_T \Gamma^B_{BC} T_{A_1 A_2 \dots A_n} + \sum_{i=1}^n \Gamma_{CA_i}{}^B T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}, \quad (3.56)$$

and its connection, or the DFT extension of the Christoffel symbol, is given by

$$\begin{aligned} \Gamma_{CAB} = \Gamma_{C[AB]} = & 2(P\partial_C P\bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E) \partial_D P_{EC} \\ & - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}{}^D + P_{C[A} P_{B]}{}^D) (\partial_D d + (P\partial^E P\bar{P})_{[ED]}). \end{aligned} \quad (3.57)$$

This expression is uniquely fixed by requiring (i) the compatibility with the DFT-dilaton and the projectors,

$$\nabla_A d = -\frac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \frac{1}{2} \Gamma^B_{BA} = 0, \quad \nabla_A P_{BC} = 0, \quad \nabla_A \bar{P}_{BC} = 0, \quad (3.58)$$

(ii) a cyclic property,

$$\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0, \quad (3.59)$$

and (iii) the kernel conditions for the six-index projectors,

$$\mathcal{P}_{ABC}{}^{DEF} \Gamma_{DEF} = 0, \quad \bar{\mathcal{P}}_{ABC}{}^{DEF} \Gamma_{DEF} = 0. \quad (3.60)$$

The cyclic property (3.59) corresponds to a torsionless condition, as it ensures that we can freely replace the ordinary derivatives in the definition of the generalized Lie derivative (3.5) by the semi-covariant derivatives,¹²

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla). \quad (3.61)$$

¹²In this work we focus on the above torsionless connection (3.57). Yet, in the ‘full order’ supersymmetric double field theories, in order to ensure the ‘1.5 formalism’, it is necessary to relax (3.59) and include torsions which are quadratic in fermions [35, 36].

In general, the semi-covariant derivative by itself is not completely covariant under DFT-diffeomorphisms.¹³ There is a potential discrepancy between the actual diffeomorphic transformation and the generalized Lie derivative of the semi-covariant derivative,

$$(\delta_X - \hat{\mathcal{L}}_X) \nabla_C T_{A_1 \dots A_n} = \sum_{i=1}^n 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}{}^{BDEF} (\partial_D \partial_E X_F) T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n} . \quad (3.62)$$

However, the diffeomorphic anomaly on the right hand side of the above equality is organized in terms of the six-index projectors, and hence it can be projected out. This explains the notion, ‘semi-covariance’. Namely, the characteristic of the semi-covariant derivative is that, it can be completely covariantized through appropriate contractions with the projectors or the DFT-vielbein [27, 28]. The *completely covariant derivatives*, relevant to the present work, are from [28],¹⁴

$$\begin{aligned} P_C{}^D \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \dots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n} , & \quad \bar{P}_C{}^D P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \dots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n} , \\ P^{AB} \bar{P}_{C_1}{}^{D_1} \bar{P}_{C_2}{}^{D_2} \dots \bar{P}_{C_n}{}^{D_n} \nabla_A T_{B D_1 D_2 \dots D_n} , & \quad \bar{P}^{AB} P_{C_1}{}^{D_1} P_{C_2}{}^{D_2} \dots P_{C_n}{}^{D_n} \nabla_A T_{B D_1 D_2 \dots D_n} . \end{aligned} \quad (3.63)$$

The second connection in the master derivative (3.54) is the spin connection for the local Lorentz symmetry of $\mathbf{Spin}(1, 3)$,

$$\Phi_{Apq} = \Phi_{A[pq]} = V_p{}^B \nabla_A V_{Bq} = V_p{}^B (\partial_A V_{Bq} + \Gamma_{AB}{}^C V_{Cq}) . \quad (3.64)$$

It is determined by requiring that the master derivative should be compatible with the DFT-vielbein (which is HS gauge singlet),

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0 . \quad (3.65)$$

The master derivative is also compatible with the $\mathbf{O}(4, 4)$ invariant metric (2.1), the $\mathbf{Spin}(1, 3)$ Minkowskian flat metric, the gamma matrices and the charge conjugation matrix:

$$\mathcal{D}_A \mathcal{J}_{BC} = \nabla_A \mathcal{J}_{BC} = 0 , \quad \mathcal{D}_A \eta_{pq} = 0 , \quad \mathcal{D}_A (\gamma^p)^\alpha{}_\beta = 0 , \quad \mathcal{D}_A \mathbf{C}_{\alpha\beta} = 0 . \quad (3.66)$$

In particular, from the compatibility with the gamma matrices, the standard relation between the spinorial and the vectorial representations of the spin connection follows

$$\Phi_A{}^\alpha{}_\beta = \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\alpha{}_\beta . \quad (3.67)$$

¹³Nevertheless, exceptions exist which include (3.58), (3.60), (3.65) *etc.* They are completely covariant by themselves, as the anomalous terms in (3.62) vanish automatically for them.

¹⁴For the ‘updated’ full list of the completely covariant derivatives, we refer to [38] (tensorial) and [52] (spinorial).

While the spin connection, Φ_{Apq} , takes good care of the $\mathbf{Spin}(1, 3)$ local Lorentz symmetry, making the master derivative always covariant under the symmetry, it is potentially anomalous for the DFT-diffeomorphisms, like (3.62),

$$(\delta_X - \hat{\mathcal{L}}_X)\Phi_{Apq} = 2\mathcal{P}_{Apq}{}^{DEF}\partial_D\partial_E X_F. \quad (3.68)$$

In a similar fashion to (3.63), only the following modules of the spin connection are completely covariant under DFT-diffeomorphisms,

$$\bar{P}_A{}^B\Phi_{Bpq}, \quad \Phi_{A[pq}V^A{}_{r]} = \Phi_{[pqr]}, \quad \Phi_{Apq}V^{Ap} = \Phi^p{}_{pq}. \quad (3.69)$$

This implies that, acting on arbitrary $\mathbf{Spin}(1, 3)$ spinors like Ψ^α or Ψ_A^α , of which the latter carries an additional $\mathbf{O}(4, 4)$ vector index, the completely covariant ‘Dirac’ operators are restricted to be the followings [30, 31, 35, 36],

$$\gamma^p\mathcal{D}_p\Psi = \gamma^A\mathcal{D}_A\Psi, \quad \bar{P}_A{}^B\mathcal{D}_B\Psi, \quad \bar{P}_A{}^B\gamma^p\mathcal{D}_p\Psi_B, \quad \bar{P}^{AB}\mathcal{D}_A\Psi_B. \quad (3.70)$$

Explicitly, the semi-covariant master derivatives of the spinors read

$$\begin{aligned} \mathcal{D}_A\Psi(x, \zeta, \bar{\zeta}) &= \partial_A\Psi + \frac{1}{4}\Phi_{Apq}\gamma^{pq}\Psi + [\mathcal{W}_A, \Psi]_\star, \\ \mathcal{D}_A\Psi_B(x, \zeta, \bar{\zeta}) &= \partial_A\Psi_B + \Gamma_{AB}{}^C\Psi_C + \frac{1}{4}\Phi_{Apq}\gamma^{pq}\Psi_B + [\mathcal{W}_A, \Psi_B]_\star. \end{aligned} \quad (3.71)$$

Note that, since we postulate in (2.17) that the $\mathbf{Spin}(1, 3)$ local Lorentz symmetry should act only on the explicit $\mathbf{Spin}(1, 3)$ indices of V_{Ap} and Ψ^α , without rotating the spinorial coordinates, *c.f.* (2.20), (2.21), the corresponding spin connection, Φ_A , also acts on the explicit $\mathbf{Spin}(1, 3)$ indices only in (3.71) not on the internal spinorial coordinates, $\zeta^\alpha, \bar{\zeta}_\beta$.¹⁵

Finally, under the higher spin gauge symmetry (2.15),

$$\delta_{\mathcal{T}}\Psi = [\mathcal{T}, \Psi]_\star, \quad \delta_{\mathcal{T}}\Psi_A = [\mathcal{T}, \Psi_A]_\star, \quad \delta_{\mathcal{T}}\mathcal{W}_A = -\mathcal{D}_A\mathcal{T}, \quad (3.72)$$

the master derivatives are covariant by themselves,

$$\delta_{\mathcal{T}}(\mathcal{D}_A\Psi) = [\mathcal{T}, \mathcal{D}_A\Psi]_\star, \quad \delta_{\mathcal{T}}(\mathcal{D}_A\Psi_B) = [\mathcal{T}, \mathcal{D}_A\Psi_B]_\star. \quad (3.73)$$

¹⁵However, continue to read (3.126) and discussion there.

3.5 Curvature and field strength

For each gauge potential in the master derivative (2.22), we set a corresponding *semi-covariant curvature* or *semi-covariant field strength*, following [28, 29, 37, 52],

$$\begin{aligned} S_{ABCD} &:= \frac{1}{2}(R_{ABCD} + R_{CDAB} - \Gamma_{AB}^E \Gamma_{ECD}), \\ \mathcal{F}_{ABpq}^\Phi &:= \nabla_A \Phi_{Bpq} - \nabla_B \Phi_{Apq} + \Phi_{Ap}{}^r \Phi_{Brq} - \Phi_{Bp}{}^r \Phi_{Arq}, \\ \mathcal{F}_{AB}^\mathcal{W} &:= \nabla_A \mathcal{W}_B - \nabla_B \mathcal{W}_A + [\mathcal{W}_A, \mathcal{W}_B]_\star, \end{aligned} \quad (3.74)$$

in which R_{ABCD} is given by

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}^E \Gamma_{BED} - \Gamma_{BC}^E \Gamma_{AED}. \quad (3.75)$$

By construction, and also due to the section condition, S_{ABCD} satisfies various identities [28],

$$\begin{aligned} S_{ABCD} &= S_{CDAB} = S_{[AB][CD]}, & S_{ABCD} + S_{BCAD} + S_{CABD} &= 0, \\ (P^{AB} P^{CD} + \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} &= 0, & P_A{}^C \bar{P}_B{}^D (P^{EF} - \bar{P}^{EF}) S_{CEDF} &= 0, \\ P_A{}^E P_B{}^F \bar{P}_C{}^G \bar{P}_D{}^H S_{EFGH} &= 0, & P_A{}^E \bar{P}_B{}^F P_C{}^G \bar{P}_D{}^H S_{EFGH} &= 0. \end{aligned} \quad (3.76)$$

Despite of all these nice properties, S_{ABCD} is not a DFT-diffeomorphism covariant tensor. It is widely speculated that there is no completely covariant four-index curvature in double field theory which can be constructed out of the geometric objects only, *i.e.* the DFT-dilaton and the projectors, *c.f.* [28, 59]. Yet, like the semi-covariant derivative (3.62), the diffeomorphic anomaly of S_{ABCD} is governed by the six-index projectors:

$$(\delta_X - \hat{\mathcal{L}}_X) S_{ABCD} = 2 \nabla_{[A} \left((\mathcal{P} + \bar{\mathcal{P}})_{B][CD]}{}^{EFG} \partial_E \partial_F X_G \right) + 2 \nabla_{[C} \left((\mathcal{P} + \bar{\mathcal{P}})_{D][AB]}{}^{EFG} \partial_E \partial_F X_G \right). \quad (3.77)$$

Thus, once again after being properly contracted with the projectors, it can produce completely covariant curvatures, such as two-index ‘Ricci-type’ curvature and ‘scalar’ curvature [28]:

$$P_A{}^C \bar{P}_B{}^D S_{CD}, \quad P^{AB} P^{CD} S_{ACBD} = -\bar{P}^{AB} \bar{P}^{CD} S_{ACBD}, \quad (3.78)$$

for which we set

$$S_{AB} = S_{BA} := S_{ACB}{}^C. \quad (3.79)$$

As Φ_{Apq} is related to Γ_{ABC} (3.64), so are their curvatures [52]. For example, we note

$$\bar{P}_A{}^B S_{Bpqr} = \frac{1}{2} \bar{P}_A{}^B \mathcal{F}_{Bpqr}^\Phi, \quad (3.80)$$

which is however not a completely covariant tensor. As for the completely covariant curvatures (3.78), we have

$$\bar{P}_A{}^B S_{Bp} = \bar{P}_A{}^B \mathcal{F}_{Bqp}^\Phi = \bar{P}_A{}^B V^{Cq} \mathcal{F}_{BCpq}^\Phi, \quad S_{pq}{}^{pq} = P^{AB} P^{CD} S_{ACBD} = \mathcal{F}_{pq}^\Phi{}^{pq} + \frac{1}{2} \Phi_{Epq} \Phi^{Epq}. \quad (3.81)$$

In a similar fashion to (3.77), $\mathcal{F}_{AB}^\mathcal{W}$ is anomalous under DFT-diffeomorphisms and also HS gauge symmetry,

$$\begin{aligned} \delta_X \mathcal{F}_{AB}^\mathcal{W} &= \hat{\mathcal{L}}_X(\mathcal{F}_{AB}^\mathcal{W}) - 2(\mathcal{P} + \bar{P})^C{}_{AB}{}^{DEF} \partial_D \partial_{[E} X_{F]} \mathcal{W}_C, \\ \delta_{\mathcal{T}} \mathcal{F}_{AB}^\mathcal{W} &= [\mathcal{T}, \mathcal{F}_{AB}^\mathcal{W}]_\star + \Gamma^C{}_{AB} \partial_C \mathcal{T}. \end{aligned} \quad (3.82)$$

The completely covariant field strength is then given by, *c.f.* [29, 37],

$$(P\mathcal{F}^\mathcal{W}\bar{P})_{AB} = P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}^\mathcal{W} = P_A{}^C \bar{P}_B{}^D (\nabla_C \mathcal{W}_D - \nabla_D \mathcal{W}_C + [\mathcal{W}_C, \mathcal{W}_D]_\star). \quad (3.83)$$

As the two $\mathbf{O}(4, 4)$ indices of $\mathcal{F}_{CD}^\mathcal{W}$ are projected orthogonally, the above quantity is clearly covariant under DFT-diffeomorphisms. Further, the section condition implies

$$P_A{}^C \bar{P}_B{}^D \Gamma^E{}_{CD} \partial_E = (P\partial^E P\bar{P})_{AB} \partial_E = 0, \quad (3.84)$$

which immediately implies, with (3.82), that $(P\mathcal{F}^\mathcal{W}\bar{P})_{AB}$ is covariant under higher spin gauge symmetry as well [29]: along with (3.72) and (3.73), we have

$$\delta_{\mathcal{T}}(P\mathcal{F}^\mathcal{W}\bar{P})_{AB} = [\mathcal{T}, (P\mathcal{F}^\mathcal{W}\bar{P})_{AB}]_\star. \quad (3.85)$$

It is worth while to note that successive applications of the completely covariant Dirac operators can produce the completely covariant ‘Ricci-type’ curvature as well as the completely covariant field strength. From the commutator relation,

$$[\mathcal{D}_A, \mathcal{D}_B] \Psi = \frac{1}{4} \mathcal{F}_{ABpq}^\Phi \gamma^{pq} \Psi + [\mathcal{F}_{AB}^\mathcal{W}, \Psi]_\star - \Gamma^C{}_{AB} \partial_C \Psi, \quad (3.86)$$

one can derive, *c.f.* [52, 60],

$$[\gamma^p \mathcal{D}_p, \bar{P}_A{}^B \mathcal{D}_B] \Psi = \frac{1}{2} \bar{P}_A{}^B S_{Bp} \gamma^p \Psi + [(P\mathcal{F}^\mathcal{W}\bar{P})_{BA}, \gamma^B \Psi]_\star, \quad (3.87)$$

in which each term on the left and the right hand sides of the equality is, from (3.70), completely covariant. Similarly, we may also obtain

$$\begin{aligned}
& (\gamma^p \mathcal{D}_p)^2 \Psi + \bar{P}^{AB} \mathcal{D}_A \mathcal{D}_B \Psi + \frac{1}{4} S_{pq}{}^{pq} \Psi \\
&= \frac{1}{2} [\mathcal{F}_{pq}^{\mathcal{W}}, \gamma^{pq} \Psi]_{\star} + 2 [\mathcal{W}_A, \mathcal{D}^A \Psi]_{\star} + [\mathcal{D}_A \mathcal{W}^A, \Psi]_{\star} - [\mathcal{W}_A, [\mathcal{W}^A, \Psi]_{\star}]_{\star} \\
&= \frac{1}{2} [\mathcal{F}_{pq}^{\mathcal{W}} + \Phi_{Apq} \mathcal{W}^A, \gamma^{pq} \Psi]_{\star} + [\mathcal{W}^A, \partial_A \Psi + [\mathcal{W}_A, \Psi]_{\star}]_{\star} + \partial_A ([\mathcal{W}^A, \Psi]_{\star}) .
\end{aligned} \tag{3.88}$$

Again, each term on the first line is completely covariant, but the other lines are covariant only as a whole expression. Yet, if Ψ were higher spin gauge singlet, they would vanish trivially and this would be consistent with the known result [52, 60].

3.6 The action and the BPS equations: DFT generalization of the Vasiliev equations

Our proposed HS-DFT Lagrangian (2.29) consists of two parts: \mathcal{L}_{DFT} for the ‘pure’ DFT Lagrangian and \mathcal{L}_{HS} for the ‘matter’ HS Lagrangian. We recall them here,

$$\mathcal{L}_{\text{DFT}} = e^{-2d} \left[(P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD} - 2\Lambda_{\text{DFT}} \right], \tag{3.89}$$

$$\mathcal{L}_{\text{HS}} = g_{\text{HS}}^{-2} e^{-2d} \text{Tr} \left[P^{AC} \bar{P}^{BD} \mathcal{F}_{AB}^{\mathcal{W}} \star \mathcal{F}_{CD}^{\mathcal{W}} + \bar{\Psi} \star \gamma^{(5)} \gamma^A \mathcal{D}_A \Psi - V_{\star}(\Psi) \right]. \tag{3.90}$$

In our proposal, we identify the genuine DFT fields, *i.e.* d and V_{Ap} , as the geometric objects, while the **HS**(4)-valued fields, *i.e.* \mathcal{W}_A and Ψ^{α} , are viewed as the matter living on the background the DFT geometry provides. As stated previously in section 2, Λ_{DFT} denotes the DFT version of the cosmological constant [28], which naturally arises in the Scherk-Schwarz reduction of the $D=10$ half-maximal super-symmetric double field theory with the ‘relaxed’ section condition on the twisting matrix [61–64], [52]. In particular, we can choose the sign of Λ_{DFT} freely, either positive or negative [52]. On account of the cosmological constant, we let the minimum of the potential vanish, *i.e.* $\min[V_{\star}(\Psi)] = 0$. The trace in (3.90) stands for the $(\zeta, \bar{\zeta})$ -integrals (2.34) over the (real eight-dimensional) internal space. This definition of the trace is formal in the sense that we do not discuss subtle issues such as its convergence or the functional class on which it is well defined.¹⁶ The only property which we actually make use of is that the trace of the star commutator vanishes, neglecting boundary terms, such that

$$\text{Tr} [f \star g] = \text{Tr} [g \star f]. \tag{3.91}$$

In order to derive the full set of the equations of motion, we need to consider the arbitrary variations of all the elementary fields, *i.e.* δd , δV_{Ap} , $\delta \mathcal{W}_A$ and $\delta \Psi^{\alpha}$. Due to the defining property of the DFT-vielbein (2.11), δV_{Ap} is constrained to meet [35]

$$\delta V_{Ap} = \bar{P}_A{}^B \delta V_{Bp} + \delta V_{B[p} V_{q]}^B V_A{}^q, \tag{3.92}$$

¹⁶Such issues become important when solving Vasiliev equations and defining observables in Vasiliev theory but are beyond the scope of this paper.

such that

$$V_{Ap}\delta V^A_q = V_{A[p}\delta V^A_{q]}, \quad (3.93)$$

and the variation of V_{Ap} is generated by an arbitrary 4×4 skew-symmetric matrix, $\Xi_{pq} = -\Xi_{qp}$, and a \bar{P} -projected 8×4 matrix, Δ_{Ap} :

$$\delta V_{Ap} = \bar{P}_A{}^B \Delta_{Bp} + V_A{}^q \Xi_{[pq]}. \quad (3.94)$$

All together, there are $6 + 16$ independent degrees of freedom in δV_{Ap} which match those of $\delta B_{\mu\nu}$ and $\delta e_\mu{}^p$ of the Riemannian parametrization (3.46).

The induced transformation of the pure DFT Lagrangian is rather well known, for which it is useful to note that the induced variation of the semi-covariant four-index curvature is ‘total derivatives’,

$$\delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]CD} + \nabla_{[C} \delta \Gamma_{D]AB}. \quad (3.95)$$

Up to total derivatives (\simeq), we have, *c.f. e.g.* [35, 36],

$$\delta \mathcal{L}_{\text{DFT}} \simeq -2 \delta d \mathcal{L}_{\text{DFT}} + 4e^{-2d} \delta V^{Ap} \bar{P}_A{}^B S_{pB}. \quad (3.96)$$

We turn to look for the variation of the higher spin ‘matter’ Lagrangian, $\delta \mathcal{L}_{\text{HS}}$. Firstly, the induced transformation of the spin connection is [35]

$$\delta \Phi_{Apq} = \mathcal{D}_A(V^B_p \delta V_{Bq}) + V^B_p V^C_q \delta \Gamma_{ABC}, \quad (3.97)$$

where and also in (3.95), $\delta \Gamma_{ABC}$ denotes the induced variation of the DFT-Christoffel connection. While it can be expressed explicitly in terms of δV_{Ap} and δd [28], for the present purpose of deriving the equations of motion, the concrete expression is luckily unnecessary: from (3.59) and the triviality of the trace of a star commutator (3.91), we note

$$\begin{aligned} & \delta \Gamma_{ABC} \text{Tr} (\bar{\Psi} \star \gamma^{(5)} \gamma^A \gamma^{BC} \Psi) \\ &= \delta \Gamma_{[ABC]} \text{Tr} (\bar{\Psi} \star \gamma^{(5)} \gamma^{ABC} \Psi) + P^{AB} \delta \Gamma_{ABC} (\mathbf{C} \gamma^{(5)} \gamma^C)_{\alpha\beta} \text{Tr} ([\Psi^\alpha, \Psi^\beta]_\star) \\ &= 0. \end{aligned} \quad (3.98)$$

We proceed to obtain through straightforward computations, up to total derivatives,¹⁷

$$\begin{aligned} \frac{1}{4} \mathcal{D}_A(V^B_p \delta V_{Bq}) \text{Tr} [\bar{\Psi} \star \gamma^{(5)} \gamma^A \gamma^{pq} \Psi] &\simeq -\frac{1}{2} \delta V_{Bq} \text{Tr} [\bar{\Psi} \star \gamma^{(5)} \gamma^{ABq} \mathcal{D}_A \Psi] \\ &\simeq \text{Tr} [\delta V_A{}^p \bar{\Psi} \star \gamma^{(5)} \gamma^A \mathcal{D}_p \Psi - \frac{1}{2} \delta V_{Ap} \bar{\Psi} \gamma^{Ap} \star \gamma^{(5)} \gamma^B \mathcal{D}_B \Psi], \end{aligned} \quad (3.99)$$

¹⁷*c.f.* appendix of [36] for the case of fermionic dilatinos.

and hence

$$\begin{aligned} & \delta \text{Tr} \left[\bar{\Psi} \star \gamma^{(5)} \gamma^A \mathcal{D}_A \Psi \right] \\ & \simeq \text{Tr} \left[\bar{P}^{AB} \delta V_{Ap} \bar{\Psi} \star \gamma^{(5)} \gamma^p \mathcal{D}_B \Psi + 2(\delta \bar{\Psi} - \frac{1}{4} \delta V_{Ap} \bar{\Psi} \gamma^{Ap}) \star \gamma^{(5)} \gamma^B \mathcal{D}_B \Psi - 2\delta \mathcal{W}_A \star \bar{\Psi} \star \gamma^{(5)} \gamma^A \Psi \right]. \end{aligned} \quad (3.100)$$

Further, from [40] (section 3.3 therein), we have

$$\begin{aligned} & \delta \text{Tr} \left[P^{AC} \bar{P}^{BD} \mathcal{F}_{AB}^{\mathcal{W}} \star \mathcal{F}_{CD}^{\mathcal{W}} \right] \\ & \simeq 4 \text{Tr} \left[\delta \mathcal{W}_A \star \mathcal{D}_B (P \mathcal{F}^{\mathcal{W}} \bar{P})^{[AB]} \right] \\ & \quad + 2V^{Ap} \delta V_p^B \text{Tr} \left[\{ P \mathcal{F}^{\mathcal{W}} (P - \bar{P}) \star \mathcal{F}^{\mathcal{W}} \bar{P} \}_{AB} + \nabla_C \{ (P \mathcal{F}^{\mathcal{W}} \bar{P})_{AB} \star \mathcal{W}^C \} \right]. \end{aligned} \quad (3.101)$$

Combining (3.100) and (3.101), we acquire

$$\begin{aligned} \delta \mathcal{L}_{\text{HS}} \simeq & -2\delta d \mathcal{L}_{\text{HS}} + 2g_{\text{HS}}^{-2} e^{-2d} \text{Tr} \left[(\delta \bar{\Psi} - \frac{1}{4} \delta V_{Ap} \bar{\Psi} \gamma^{Ap}) \star \gamma^{(5)} \gamma^B \mathcal{D}_B \Psi - \frac{1}{2} \delta \bar{\Psi} \star \mathbf{C}^{-1} \partial_\Psi V_\star(\Psi) \right] \\ & + 4g_{\text{HS}}^{-2} e^{-2d} \text{Tr} \left[\delta \mathcal{W}_A \star \{ \mathcal{D}_B (P \mathcal{F}^{\mathcal{W}} \bar{P})^{[AB]} - \frac{1}{2} \bar{\Psi} \star \gamma^{(5)} \gamma^A \Psi \} \right] \\ & + 2g_{\text{HS}}^{-2} e^{-2d} V^{Ap} \delta V_p^B \text{Tr} \left[\{ P \mathcal{F}^{\mathcal{W}} (P - \bar{P}) \star \mathcal{F}^{\mathcal{W}} \bar{P} \}_{AB} + \nabla_C \{ (P \mathcal{F}^{\mathcal{W}} \bar{P})_{AB} \star \mathcal{W}^C \} \right. \\ & \quad \left. + \frac{1}{2} \bar{P}_B^C \bar{\Psi} \star \gamma^{(5)} \gamma_A \mathcal{D}_C \Psi \right]. \end{aligned} \quad (3.102)$$

The full Euler-Lagrange equations then follow from (3.96) and (3.102): for the DFT-dilaton,

$$\mathcal{L}_{\text{HS-DFT}} = \mathcal{L}_{\text{DFT}} + \mathcal{L}_{\text{HS}} = 0; \quad (3.103)$$

for the DFT-vielbein from the variation, $\delta V_{Ap} = V_A^q \Xi_{[pq]}$ (3.94),

$$\bar{\Psi} \star \gamma^{pq} \gamma^{(5)} \gamma^A \mathcal{D}_A \Psi = 0; \quad (3.104)$$

for the DFT-vielbein from the variation, $\delta V_{Ap} = \bar{P}_A^B \Delta_{Bp}$,¹⁸

$$P_A^C \bar{P}_B^D \left(S_{CD} + \frac{1}{2} g_{\text{HS}}^{-2} \text{Tr} \left[\{ \mathcal{F}^{\mathcal{W}} \star (P - \bar{P}) \mathcal{F}^{\mathcal{W}} \}_{CD} + \nabla_E (\mathcal{F}_{CD}^{\mathcal{W}} \star \mathcal{W}^E) + \frac{1}{2} \bar{\Psi} \star \gamma^{(5)} \gamma_C \mathcal{D}_D \Psi \right] \right) = 0; \quad (3.105)$$

¹⁸Though not thoroughly written in terms of $(P \mathcal{F}^{\mathcal{W}} \bar{P})_{AB}$ (3.83), it is straightforward to show from (3.82) that the expression of (3.105) is completely covariant under both DFT-diffeomorphisms and HS gauge symmetry, *c.f.* Ref. [40].

for the HS gauge potential,

$$\mathcal{D}_B (P\mathcal{F}^{\mathcal{W}}\bar{P})^{[AB]} - \frac{1}{2}\bar{\Psi} \star \gamma^{(5)}\gamma^A \Psi = 0; \quad (3.106)$$

and lastly for the bosonic **Spin**(1, 3) Majorana spinor,

$$\gamma^A \mathcal{D}_A \Psi - \frac{1}{2}\gamma^{(5)}\mathbf{C}^{-1}\partial_\Psi V_\star(\Psi) = 0, \quad (3.107)$$

which actually implies (3.104) provided the potential, $V_\star(\Psi)$, is **Spin**(1, 3) singlet.

From the orthogonality, $\bar{P}_A{}^B V_{Bp} = 0$, the equation of motion of the HS gauge potential (3.106) actually implies both

$$\mathcal{D}_B (P\mathcal{F}^{\mathcal{W}}\bar{P})^{BA} = 0, \quad (3.108)$$

and

$$\mathcal{D}_B (P\mathcal{F}^{\mathcal{W}}\bar{P})^{AB} - \bar{\Psi} \star \gamma^{(5)}\gamma^A \star \Psi = 0. \quad (3.109)$$

Further, the skew-symmetric property of $(\mathbf{C}\gamma^{(5)}\gamma^p)_{\alpha\beta}$ (2.4) gives an identity,

$$\bar{\Psi} \star \gamma^{(5)}\gamma^p \Psi = \frac{1}{2} \left[\Psi^\alpha, \Psi^\beta \right]_\star (\mathbf{C}\gamma^{(5)}\gamma^p)_{\alpha\beta}, \quad (3.110)$$

and hence we may rewrite (3.109) as

$$\mathcal{D}_B (P\mathcal{F}^{\mathcal{W}}\bar{P})^{AB} - \frac{1}{2} \left[\Psi^\alpha, \Psi^\beta \right]_\star (\mathbf{C}\gamma^{(5)}\gamma^A)_{\alpha\beta} = 0. \quad (3.111)$$

Clearly, all the equations of motion, (3.103), (3.104), (3.105), (3.107), (3.108), (3.109) are automatically fulfilled, if we assume the DFT equations of motion, (2.42), (2.43),

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda_{\text{DFT}} - g_{\text{HS}}^{-2}\text{Tr}[V_\star(\Psi)] = 0, \quad (3.112)$$

$$P_A{}^C \bar{P}_B{}^D S_{CD} = 0, \quad (3.113)$$

and the first or zeroth order differential BPS equations, (2.44)–(2.48),

$$P_A{}^C \bar{P}_B{}^D \mathcal{F}_{CD}^{\mathcal{W}}(x, \zeta, \bar{\zeta}) = 0, \quad (3.114)$$

$$\bar{P}_A{}^B \mathcal{D}_B \Psi(x, \zeta, \bar{\zeta}) = 0, \quad (3.115)$$

$$\gamma^A \mathcal{D}_A \Psi(x, \zeta, \bar{\zeta}) = 0, \quad (3.116)$$

$$\left[\Psi^\alpha(x, \zeta, \bar{\zeta}), \Psi^\beta(x, \zeta, \bar{\zeta}) \right]_\star (\mathbf{C}\gamma^{(5)}\gamma^p)_{\alpha\beta} = 0, \quad (3.117)$$

$$\partial_\Psi \text{Tr}[V_\star(\Psi)] = 0. \quad (3.118)$$

It is worth while to note, from the commutator relation (3.87) that, there is a mutual consistency among (3.113), (3.114), (3.115) and (3.116). Furthermore, ignoring the adjoint action of the HS gauge potential, the two conditions, (3.115) and (3.116), are precisely the supersymmetry transformations of the gravitino and the dilatino respectively in the half-maximal supersymmetric double field theory [35] (see also [60]). This also supports our nomenclature, ‘BPS’.

From the 4×4 skew-symmetric completeness relation (2.6), the algebraic commutator relation (3.117) is equivalent to

$$[\Psi^\alpha, \Psi^\beta]_\star = -\frac{1}{2}(\bar{\Psi} \star \Psi) \mathbf{C}^{-1\alpha\beta} - \frac{1}{2}(\bar{\Psi} \star \gamma^{(5)} \star \Psi)(\gamma^{(5)} \mathbf{C}^{-1})^{\alpha\beta}. \quad (3.119)$$

Thus, if we let

$$\mathcal{Q}_+ := -\bar{\Psi} \star (1 + \gamma^{(5)}) \star \Psi, \quad \mathcal{Q}_- := -\bar{\Psi} \star (1 - \gamma^{(5)}) \star \Psi, \quad (3.120)$$

satisfying the reality relation,

$$\mathcal{Q}_+ = (\mathcal{Q}_-)^{\dagger}, \quad (3.121)$$

(3.119) can be rewritten as (2.49), *i.e.*

$$[\Psi^\alpha, \Psi^\beta]_\star = \frac{1}{4}[(1 + \gamma^{(5)}) \mathbf{C}^{-1}]^{\alpha\beta} \mathcal{Q}_+ + \frac{1}{4}[(1 - \gamma^{(5)}) \mathbf{C}^{-1}]^{\alpha\beta} \mathcal{Q}_-. \quad (3.122)$$

We recall the expansion of the Majorana spinor field, Ψ^α , in terms of the internal spinorial coordinates (3.17),

$$\Psi^\alpha(x, \zeta, \bar{\zeta}) = \sum_{m,n} \frac{1}{m!n!} \zeta^{\alpha_1} \zeta^{\alpha_2} \dots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \dots \bar{\zeta}_{\beta_n} \Psi^\alpha_{\alpha_1 \alpha_2 \dots \alpha_m}{}^{\beta_1 \beta_2 \dots \beta_n}(x), \quad (3.123)$$

and decompose the HS gauge potential as

$$\mathcal{W}_A = \frac{1}{4} \Phi_{Apq} \bar{\zeta} \gamma^{pq} \zeta + \mathcal{W}'_A. \quad (3.124)$$

Then, the master derivative of the spinor field gives, with (3.39),

$$\begin{aligned} \mathcal{D}_A \Psi^\alpha(x, \zeta, \bar{\zeta}) &= \partial_A \Psi^\alpha(x, \zeta, \bar{\zeta}) + \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\alpha{}_\beta \Psi^\beta(x, \zeta, \bar{\zeta}) + [\mathcal{W}_A, \Psi^\alpha(x, \zeta, \bar{\zeta})]_\star \\ &= \sum_{m,n} \frac{1}{m!n!} \left\{ \zeta^{\alpha_1} \zeta^{\alpha_2} \dots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \dots \bar{\zeta}_{\beta_n} \mathcal{D}'_A \Psi^\alpha_{\alpha_1 \alpha_2 \dots \alpha_m}{}^{\beta_1 \beta_2 \dots \beta_n}(x) \right. \\ &\quad \left. + [\mathcal{W}'_A, \zeta^{\alpha_1} \zeta^{\alpha_2} \dots \zeta^{\alpha_m} \bar{\zeta}_{\beta_1} \bar{\zeta}_{\beta_2} \dots \bar{\zeta}_{\beta_n}]_\star \Psi^\alpha_{\alpha_1 \alpha_2 \dots \alpha_m}{}^{\beta_1 \beta_2 \dots \beta_n}(x) \right\}. \end{aligned} \quad (3.125)$$

In the above, we set for the component field,

$$\begin{aligned} \mathcal{D}'_A \Psi^\alpha_{\alpha_1 \dots \alpha_m}{}^{\beta_1 \dots \beta_n} &\equiv \partial_A \Psi^\alpha_{\alpha_1 \alpha_2 \dots \alpha_m}{}^{\beta_1 \beta_2 \dots \beta_n} + \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\alpha{}_\beta \Psi^\beta_{\alpha_1 \dots \alpha_m}{}^{\beta_1 \dots \beta_n} \\ &\quad - \sum_{i=1}^m \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\gamma{}_{\alpha_i} \Psi^\alpha_{\alpha_1 \dots \gamma \dots \alpha_m}{}^{\beta_1 \dots \beta_n} + \sum_{j=1}^n \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^{\beta_j}{}_\delta \Psi^\alpha_{\alpha_1 \dots \alpha_m}{}^{\beta_1 \dots \delta \dots \beta_n}. \end{aligned} \quad (3.126)$$

This is a suggestive form, especially with respect to the possible modification of the **Spin**(1, 3) local Lorentz transformation rule as (2.20) and (2.21), since the spin connection now acts equally on all the spinorial indices of the component fields.

3.7 Linear DFT-dilaton vacuum

To solve for a vacuum solution, we make an ansatz to put, with (3.14), (3.124),

$$\mathring{\Psi}^\alpha \equiv m^{\frac{3}{2}} \mathbf{Re}(\zeta^\alpha) = \frac{1}{2} m^{\frac{3}{2}} (\zeta^\alpha + \bar{\zeta}_\beta \mathbf{C}^{-1\beta\alpha}), \quad \mathring{\mathcal{W}}'_A \equiv 0, \quad (3.127)$$

satisfying, from (3.125),

$$\mathring{\mathcal{W}}_A = \frac{1}{4} \mathring{\Phi}_{Apq} \bar{\zeta} \gamma^{pq} \zeta, \quad \mathring{\mathcal{D}}_A \mathring{\Psi}^\alpha = 0, \quad [\mathring{\Psi}^\alpha, \mathring{\Psi}^\beta]_\star = \frac{1}{2} m^3 \mathbf{C}^{-1\alpha\beta}, \quad (3.128)$$

of which the last commutator relation implies

$$\mathring{\mathcal{Q}}_+ = \mathring{\mathcal{Q}}_- = m^3. \quad (3.129)$$

Hereafter, the circle ‘ \circ ’ denotes the vacuum. Since the HS gauge potential is a DFT-diffeomorphism covariant vector but the spin connection is not, *c.f.* (3.68), we are now looking for a vacuum configuration which breaks DFT-diffeomorphisms spontaneously. As is the case with the potentials, (2.58) and (2.59), we also assume the potential to take the absolute minimum value, $V_\star(\mathring{\Psi}) = \min[V_\star] = 0$, when $\Psi = \mathring{\Psi}$.

From (3.74) and (3.80), it is straightforward to obtain

$$\mathring{\mathcal{F}}_{AB}^{\mathring{\mathcal{W}}} = \frac{1}{4} \mathring{\mathcal{F}}_{ABpq}^{\mathring{\Phi}} \bar{\zeta} \gamma^{pq} \zeta, \quad (3.130)$$

and

$$\mathring{P}_A^C \mathring{P}_B^D \mathring{\mathcal{F}}_{CD}^{\mathring{\mathcal{W}}} = \frac{1}{2} \mathring{P}_A^C \mathring{P}_B^D \mathring{S}_{CDpq} \bar{\zeta} \gamma^{pq} \zeta. \quad (3.131)$$

Then, from the identities (3.76), including

$$\mathring{P}_A^C \mathring{P}_B^D \mathring{S}_{CEDF} \mathring{P}^{EF} = \mathring{P}_A^C \mathring{P}_B^D \mathring{S}_{CEDF} \mathring{P}^{EF} = \frac{1}{2} \mathring{P}_A^C \mathring{P}_B^D \mathring{S}_{CD}, \quad (3.132)$$

we note that (3.114) implies (3.113). Thus, with (3.112) and (3.131), the remaining conditions to fulfill all the HS-DFT BPS equations are

$$\mathring{P}_A{}^B \mathring{S}_{Bpqr} = 0, \quad (3.133)$$

and

$$(\mathring{P}^{AB} \mathring{P}^{CD} - \mathring{P}^{AB} \mathring{P}^{CD}) \mathring{S}_{ACBD} - 2\Lambda_{\text{DFT}} = 0. \quad (3.134)$$

Remarkably, as we show shortly, backgrounds with linear DFT-dilaton and constant DFT-vielbein can satisfy these two conditions,

$$\mathring{d} = \mathring{N}_A x^A, \quad \partial_A \mathring{V}_{Bp} = 0. \quad (3.135)$$

Here we have parametrized the linear DFT-dilaton by an $\mathbf{O}(D, D)$ constant vector, \mathring{N}^A , which should have the mass dimension. Though our main interest lies in the case of $D = 4$, as our discussion holds in arbitrary spacetime dimensions, we keep D free for a while. Since \mathring{N}_A is given by the partial derivative of the DFT-dilaton,

$$\mathring{N}_A = \partial_A \mathring{d}, \quad (3.136)$$

the constant $\mathbf{O}(D, D)$ vector must be null and satisfies the ‘section condition’ for consistency,

$$\mathring{N}^A \mathring{N}_A = \partial^A \mathring{d} \partial_A \mathring{d} = 0, \quad \mathring{N}^A \partial_A = \partial^A \mathring{d} \partial_A = 0. \quad (3.137)$$

The corresponding DFT-Christoffel connection and the spin connection are all constant,

$$\mathring{\Gamma}_{CAB} = -\frac{4}{D-1} \left(\mathring{P}_{C[A} \mathring{P}_{B]}^D + \mathring{P}_{C[A} \mathring{P}_{B]}^D \right) \mathring{N}_D, \quad \mathring{\Phi}_{Apq} = -\frac{4}{D-1} \mathring{V}_{A[p} \mathring{N}_{q]}. \quad (3.138)$$

In particular, from (3.128), we have

$$\mathring{W}_A = -\frac{1}{D-1} \bar{\zeta} \gamma_{Ap} \zeta \mathring{N}^p, \quad \mathring{P}_A{}^B \mathring{\Phi}_{Bpq} = 0, \quad \mathring{\Phi}_{[pqr]} = 0, \quad \mathring{\Phi}_{pq}^p = -2\mathring{N}_q. \quad (3.139)$$

Finally, from

$$\mathring{R}_{CDAB} = \frac{16}{(D-1)^2} \left[\mathring{P}_{A[D} (\mathring{P} \mathring{N})_{E]} \mathring{P}_{B[C} (\mathring{P} \mathring{N})_{F]} + \mathring{P}_{A[D} (\mathring{P} \mathring{N})_{E]} \mathring{P}_{B[C} (\mathring{P} \mathring{N})_{F]} - (A \leftrightarrow B) \right] \mathcal{J}^{EF}, \quad (3.140)$$

we confirm that the linear DFT-dilaton background indeed solves (3.133),

$$\mathring{P}_A{}^B \mathring{S}_{Bpqr} = 0, \quad (3.141)$$

giving

$$\mathring{P}_A{}^C \mathring{P}_B{}^D \mathring{S}_{CD} = 0, \quad (3.142)$$

and produces a constant scalar curvature,

$$\mathring{S}_{pq}{}^{pq} = \mathring{P}^{AB} \mathring{P}^{CD} \mathring{S}_{ACBD} = -\mathring{P}^{AB} \mathring{P}^{CD} \mathring{S}_{ACBD} = 4\mathring{P}_{AB} \mathring{N}^A \mathring{N}^B = -4\mathring{N}_p \mathring{N}^p. \quad (3.143)$$

This fulfills the remaining last condition (3.134), provided the DFT cosmological constant is matched by

$$\Lambda_{\text{DFT}} = -4\mathring{N}_p \mathring{N}^p. \quad (3.144)$$

Henceforth, we consider converting the DFT frame to the string or Einstein frames, while sacrificing the manifest $\mathbf{O}(D, D)$ symmetry. Parameterizing the DFT-vielbein and the DFT-dilaton, either in terms of the string framed fields, $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$, through (3.46), (3.47) and

$$e^{-2d} = \sqrt{-g} e^{-2\phi}, \quad (3.145)$$

or alternatively in terms of the Einstein framed fields, $\{G_{\mu\nu}, B_{\mu\nu}, \Phi\}$, by

$$\phi = \sqrt{\frac{D-2}{8}} \Phi, \quad g_{\mu\nu} = e^{\sqrt{\frac{2}{D-2}} \Phi} G_{\mu\nu}, \quad (3.146)$$

the pure DFT Lagrangian (3.89) gives, with $H = dB$ and up to total derivatives (\simeq),

$$\sqrt{-g} e^{-2\phi} \left(R_g + 4(\partial\phi)_g^2 - \frac{1}{12} H_g^2 - 2\Lambda_{\text{DFT}} \right) \simeq \sqrt{-G} \left(R_G - \frac{1}{2} (\partial\Phi)_G^2 - \frac{1}{12} e^{\sqrt{\frac{8}{D-2}} \Phi} H_G^2 - 2\Lambda_{\text{DFT}} e^{\sqrt{\frac{2}{D-2}} \Phi} \right), \quad (3.147)$$

in which the superscripts, g and G , indicate which metric is used.

Now, especially for the linear DFT-dilaton vacuum (3.135), if we assume \mathring{N}^p is a space-like D -dimensional vector and hence the DFT cosmological constant is negative,

$$\Lambda_{\text{DFT}} = -4\mathring{N}_p \mathring{N}^p < 0, \quad (3.148)$$

we may write the solution as

$$\phi = \sqrt{-\Lambda_{\text{DFT}}/2} x^{D-1}, \quad g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (3.149)$$

where x^{D-1} denotes the last spatial coordinate, $x^\mu = (x^0, x^1, \dots, x^{D-1})$.

With the dictionary (3.146), the above vacuum solution then corresponds to the known $(D-2)$ -brane background obtained in [65] in the Einstein frame: introducing one new coordinate, $x^{D-1} \rightarrow z$, satisfying

$$\sqrt{-2\Lambda_{\text{DFT}}} z = e^{\sqrt{-2\Lambda_{\text{DFT}}} x^{D-1}} \equiv K, \quad dz = K dx^{D-1}, \quad (3.150)$$

the linear DFT-dilaton vacuum in the DFT/string frame becomes the brane configuration in the Einstein frame,

$$e^\Phi = K \sqrt{\frac{2}{D-2}}, \quad G_{\mu\nu} dx^\mu dx^\nu = K^{-\frac{2}{D-2}} \eta'_{\mu'\nu'} dx^{\mu'} dx^{\nu'} + K^{-2(\frac{D-1}{D-2})} dz^2. \quad (3.151)$$

3.8 Truncation to the bosonic Vasiliev HS equations

In order to truncate our proposal of the HS-DFT BPS equations, (2.42)–(2.48), and to derive the bosonic four-dimensional Vasiliev HS equations, it is necessary to impose some extra conditions.

Firstly, we constrain the higher spin gauge potential to meet, *c.f.* [37],

$$\mathcal{W}^A \partial_A \equiv 0, \quad \mathcal{W}^A \mathcal{W}_A \equiv 0. \quad (3.152)$$

These imply with the section condition (2.7),

$$(\partial_A + \mathcal{W}_A)(\partial^A + \mathcal{W}^A) \equiv 0. \quad (3.153)$$

For consistency, under all the symmetry transformations, including the $\mathbf{O}(4, 4)$ T-duality rotations, DFT-diffeomorphisms, and the HS gauge symmetry, the above constraints (3.152) are well preserved, such as $(\delta_X \mathcal{W}^A) \partial_A \equiv 0$ (2.14), $(\delta_{\mathcal{T}} \mathcal{W}^A) \partial_A \equiv 0$ (2.15), *etc.*

We fix the section as (3.4), and consequently solve the constraints (3.152),

$$\partial_A = \left(\frac{\partial}{\partial \bar{x}_\mu}, \frac{\partial}{\partial x^\mu} \right) \equiv (0, \frac{\partial}{\partial x^\mu}), \quad \mathcal{W}_A = (\tilde{W}^\mu, W_\mu) \equiv (0, W_\mu). \quad (3.154)$$

We proceed to parametrize the DFT-dilaton and the DFT-vilebein in terms of the string framed Riemannian fields, $\{\phi, B_{\mu\nu}, e_\mu^p, g_{\mu\nu}\}$, as (3.46), (3.47) and (3.145). Then the DFT equations of motion, (2.42), (2.43), correspond to nothing but the Euler-Lagrange equations of the string framed Lagrangian of (3.147), up to the potential term:

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \partial_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} &= 0, \quad \nabla^\lambda H_{\lambda\mu\nu} - 2(\partial^\lambda \phi) H_{\lambda\mu\nu} = 0, \\ e^{+2d} \mathcal{L}_{\text{DFT}} - g_{\text{HS}}^{-2} \text{Tr} [V_\star(\Psi)] &= R + 4\Box\phi - 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - 2\Lambda_{\text{DFT}} - g_{\text{HS}}^{-2} \text{Tr} [V_\star(\Psi)] = 0. \end{aligned} \quad (3.155)$$

On the other hand, the remaining first or zeroth order differential BPS conditions, (2.44)–(2.47), become

$$\partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]_\star = 0, \quad (3.156)$$

$$\partial_\mu \Psi + \frac{1}{4} \omega_{\mu pq} \gamma^{pq} \Psi + \frac{1}{8} H_{\mu pq} \gamma^{pq} \Psi + [W_\mu, \Psi]_\star = 0, \quad (3.157)$$

$$\gamma^\mu \left(\partial_\mu \Psi + \frac{1}{4} \omega_{\mu pq} \gamma^{pq} \Psi + \frac{1}{24} H_{\mu pq} \gamma^{pq} \Psi - \partial_\mu \phi \Psi + [W_\mu, \Psi]_\star \right) = 0, \quad (3.158)$$

$$[\Psi^\alpha, \Psi^\beta]_\star = \frac{1}{4} \{ (1 + \gamma^{(5)}) \mathbf{C}^{-1} \}^{\alpha\beta} \mathcal{Q}_+ + \frac{1}{4} \{ (1 - \gamma^{(5)}) \mathbf{C}^{-1} \}^{\alpha\beta} \mathcal{Q}_-, \quad (3.159)$$

$$\partial_\Psi \text{Tr} [V_\star(\Psi)] = 0, \quad (3.160)$$

where $\omega_{\mu pq} = (e^{-1})_p{}^\nu \nabla_\mu e_{\nu q}$ denotes the standard spin connection in supergravity, *e.g.* [60], and \mathcal{Q}_\pm are defined in (2.50). Combining (3.157) and (3.158), we have an algebraic relation,

$$(\gamma^\mu \partial_\mu \phi + \frac{1}{12} \gamma^{\mu\nu\rho} H_{\mu\nu\rho}) \Psi = 0. \quad (3.161)$$

Secondly, we let the DFT-dilaton and the DFT-vielbein all trivial, or constant. This breaks the local $\mathbf{Spin}(1, 3)$ Lorentz symmetry to a global symmetry, and sets the DFT-Christoffel connection, the local Lorentz spin connection, and the semi-covariant four-index curvature all trivial,

$$\Gamma_{ABC} \equiv 0, \quad \Phi_{Apq} \equiv 0, \quad S_{ABCD} \equiv 0, \quad (3.162)$$

such that, for the Riemannian fields we have

$$\omega_{\mu pq} \equiv 0, \quad R_{\mu\nu} \equiv 0, \quad H_{\lambda\mu\nu} \equiv 0, \quad \partial_\mu \phi \equiv 0. \quad (3.163)$$

Basically, it eliminates any trace of DFT, solving the pure DFT equations of motion, (2.42) and (2.43), in a trivial manner, with the vanishing cosmological constant, $\Lambda_{\text{DFT}} \equiv 0$.

Thirdly, we assume the bosonic truncation (3.38), such that the HS gauge potential and the bosonic Majorana spinor should meet

$$W_\mu(x, -\zeta, -\bar{\zeta}) = +W_\mu(x, \zeta, \bar{\zeta}), \quad \Psi^\alpha(x, -\zeta, -\bar{\zeta}) = -\Psi^\alpha(x, \zeta, \bar{\zeta}). \quad (3.164)$$

We proceed to convert the four-component $\mathbf{Spin}(1, 3)$ spinors to two-component Weyl spinors. For this, we put the gamma five matrix into a diagonal form,

$$\gamma^{(5)} = \begin{pmatrix} \delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \\ 0 & -\delta_{\alpha}^{\beta} \end{pmatrix}, \quad (3.165)$$

and decompose the Majorana and pseudo-Majorana spinors into chiral and anti-chiral Weyl spinors,

$$\Psi^\alpha = i\frac{1}{2}m^{\frac{3}{2}} \begin{pmatrix} s^{\dot{\alpha}} \\ \bar{s}_{\dot{\alpha}} \end{pmatrix}, \quad \mathbf{Re}(\zeta) = \zeta_+^\alpha = \frac{1}{2} \begin{pmatrix} y^{\dot{\alpha}} \\ \bar{y}_{\dot{\alpha}} \end{pmatrix}, \quad \mathbf{Im}(\zeta) = \zeta_-^\alpha = \frac{1}{2} \begin{pmatrix} z^{\dot{\alpha}} \\ \bar{z}_{\dot{\alpha}} \end{pmatrix}. \quad (3.166)$$

Hereafter, the top-dotted and the bottom-dotted indices, $\dot{\alpha}, \dot{\beta} = 1, 2$ and $\alpha, \beta = 1, 2$, are the chiral and anti-chiral Weyl spinorial indices. We also employ the following explicit representation of the gamma matrices,

$$\gamma^p = \begin{pmatrix} 0 & (\sigma^p)^{\dot{\alpha}\dot{\beta}} \\ (\bar{\sigma}^p)_{\alpha\beta} & 0 \end{pmatrix}, \quad (3.167)$$

as well as the $\mathbf{A}, \mathbf{B}, \mathbf{C}$ matrices, (2.3), (2.4), (3.8),

$$\mathbf{A} = \begin{pmatrix} 0 & -i\delta_{\alpha\dot{\beta}} \\ +i\delta^{\dot{\alpha}}_{\beta} & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -\bar{\epsilon}^{-1\alpha\dot{\beta}} \\ -\epsilon_{\alpha\dot{\beta}} & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} +i\epsilon_{\alpha\dot{\beta}} & 0 \\ 0 & -i\bar{\epsilon}^{-1\alpha\dot{\beta}} \end{pmatrix}, \quad (3.168)$$

where, as 2×2 matrices, σ^0 and $-\bar{\sigma}^0$ are identity matrices,

$$\sigma^0 = -\bar{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.169)$$

$\sigma^i = \bar{\sigma}^i$ ($i = 1, 2, 3$) are the Pauli matrices,

$$\sigma^1 = \bar{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \bar{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \bar{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.170)$$

and $\epsilon_{\alpha\dot{\beta}}, \bar{\epsilon}_{\alpha\dot{\beta}}$ correspond to the usual skew-symmetric 2×2 matrices,

$$\epsilon = \bar{\epsilon} = -\epsilon^{-1} = -\bar{\epsilon}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.171)$$

satisfying

$$(\bar{\sigma}^p)_{\alpha\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\gamma}}(\sigma^p)^{\dot{\gamma}\delta}\bar{\epsilon}_{\delta\dot{\beta}}. \quad (3.172)$$

It follows that the Majorana and the pseudo-Majorana conditions, (3.11), (3.15), agree with the Vasiliev's reality conditions,

$$s_{\dot{\alpha}} = s^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = -(\bar{s}_{\alpha})^{\dagger}, \quad y_{\dot{\alpha}} = y^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = (\bar{y}_{\alpha})^{\dagger}, \quad z_{\dot{\alpha}} = z^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} = -(\bar{z}_{\alpha})^{\dagger}. \quad (3.173)$$

Basically, under the complex conjugation, the top-dotted index and the bottom-dotted index flip to each other, while their positions can be raised or lowered by ϵ , $\bar{\epsilon}$ and the inverses only.¹⁹ Further, the star commutator relations read in terms of the Weyl spinor variables, from (A.5),

$$[y_{\dot{\alpha}}, y_{\dot{\beta}}]_{\star} = +2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [\bar{y}_{\alpha}, \bar{y}_{\beta}]_{\star} = +2i\bar{\epsilon}_{\alpha\beta}, \quad [z_{\dot{\alpha}}, z_{\dot{\beta}}]_{\star} = -2i\epsilon_{\dot{\alpha}\dot{\beta}}, \quad [\bar{z}_{\alpha}, \bar{z}_{\beta}]_{\star} = -2i\bar{\epsilon}_{\alpha\beta}. \quad (3.174)$$

The remaining nontrivial HS-DFT BPS equations, (3.156), (3.157), (3.158), (3.159), then reduce to

$$\begin{aligned} \partial_{\mu}W_{\nu} - \partial_{\nu}W_{\mu} + [W_{\mu}, W_{\nu}]_{\star} &= 0, \\ \partial_{\mu}s_{\dot{\alpha}} + [W_{\mu}, s_{\dot{\alpha}}]_{\star} &= 0, \quad \partial_{\mu}\bar{s}_{\alpha} + [W_{\mu}, \bar{s}_{\alpha}]_{\star} = 0, \end{aligned} \quad (3.175)$$

$$[s_{\dot{\alpha}}, s_{\dot{\beta}}]_{\star} = -2i\epsilon_{\dot{\alpha}\dot{\beta}}m^{-3}\mathcal{Q}_{+}, \quad [s_{\dot{\alpha}}, \bar{s}_{\beta}]_{\star} = 0, \quad [\bar{s}_{\alpha}, \bar{s}_{\beta}]_{\star} = -2i\bar{\epsilon}_{\alpha\beta}m^{-3}\mathcal{Q}_{-},$$

where \mathcal{Q}_{+} and \mathcal{Q}_{-} are the quantities defined in (2.50) which now read in terms of the Weyl spinors,

$$\mathcal{Q}_{+} = i\frac{1}{2}m^3s_{\dot{\alpha}}\star s^{\dot{\alpha}}, \quad \mathcal{Q}_{-} = i\frac{1}{2}m^3\bar{s}_{\alpha}\star \bar{s}^{\alpha} = \mathcal{Q}_{+}^{\dagger}. \quad (3.176)$$

The extra conditions to fully achieve the bosonic Vasiliev HS equations are then,

$$\{s_{\dot{\alpha}}, \mathcal{Q}_{+}\}_{\star} = 2m^3s_{\dot{\alpha}}, \quad \{\bar{s}_{\alpha}, \mathcal{Q}_{-}\}_{\star} = 2m^3\bar{s}_{\alpha}, \quad (3.177)$$

and, with the inner Klein operators (3.30),

$$\mathcal{Q}_{+} - m^3 = (\mathcal{Q}_{-} - m^3)\star \mathbf{k}\star \bar{\mathbf{k}}. \quad (3.178)$$

The algebraic conditions of (3.177) are equivalent to the four-component expressions of (2.56), as well as to²⁰

$$i\frac{1}{4}s_{\dot{\alpha}}\star s_{\dot{\beta}}\star s^{\dot{\alpha}} = s_{\dot{\beta}}, \quad i\frac{1}{4}\bar{s}_{\alpha}\star \bar{s}_{\beta}\star \bar{s}^{\alpha} = \bar{s}_{\beta}. \quad (3.179)$$

¹⁹This differs from the four-component spinorial convention (3.11).

²⁰Alternative to the deformed oscillator relation (3.179), the commutator relations (2.61) become in terms of the Weyl spinors,

$$[s_{\dot{\alpha}}, s_{\dot{\beta}}s^{\dot{\beta}}] = 0, \quad [\bar{s}_{\alpha}, \bar{s}_{\beta}\bar{s}^{\beta}] = 0.$$

Further, they provide (though not the most general) solutions to the very last BPS equation (3.160), once we choose the “deformed oscillator” potential, $V_{\star}^{\text{def. osc.}}(\Psi)$ in (2.59). Especially, the cases of $\mathcal{Q}_+ = \mathcal{Q}_-$ (real) or $\mathcal{Q}_+ - m^3 = -(\mathcal{Q}_- - m^3)$ (pure imaginary) correspond to A or B model respectively, in the sense of Ref. [66]. To summarize, we have obtained the bosonic Vasiliev equations [3], written in the concise form²¹ presented in [45].

Finally, it is worth while to note that, upon the sectioning condition for the HS gauge potential (3.154), if we set ϕ and B -field trivial, our proposed HS-DFT functional (2.29) reduces to a ‘undoubled’ (Riemannian) gravity action,

$$\int d^4x e \left[R + g_{\text{HS}}^{-2} \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{\sqrt{2}} \bar{\Psi} \star \gamma^{(5)} \gamma^\mu D_\mu \Psi - V_{\star}(\Psi) \right\} \right], \quad (3.180)$$

where we set $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]_{\star}$ and $D_\mu = \nabla_\mu + \omega_\mu + W_\mu$ denotes the Riemannian master derivative, such that $D_\mu \Psi = \partial_\mu \Psi + \frac{1}{4} \omega_{\mu pq} \gamma^{pq} \Psi + [W_\mu, \Psi]_{\star}$, $D_\mu e_\nu{}^p = \nabla_\mu e_\nu{}^p + \omega_\mu{}^p{}_q e_\nu{}^q = 0$.

The remarkable fact is then that the following four ‘BPS’ conditions,

$$F_{\mu\nu} = 0, \quad D_\mu \Psi = 0, \quad [\Psi^\alpha, \Psi^\beta]_{\star} (\mathbf{C} \gamma^{(5)} \gamma^\nu)_{\alpha\beta} = 0, \quad \partial_\Psi \text{Tr} [V_{\star}(\Psi)] = 0, \quad (3.181)$$

supplemented by an Einstein manifold relation,

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} g_{\text{HS}}^{-2} \text{Tr} [V_{\star}(\Psi)], \quad (3.182)$$

can automatically solve the full set of the Euler-Lagrange equations of the above action which are explicitly,

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= \frac{1}{2} g_{\text{HS}}^{-2} \text{Tr} \left[F_{\mu\lambda} \star F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F_{\kappa\lambda} \star F^{\kappa\lambda} - \frac{1}{\sqrt{2}} \bar{\Psi} \star \gamma^{(5)} \{ \gamma_{(\mu} D_{\nu)} - g_{\mu\nu} \gamma^\lambda D_\lambda \} \Psi - g_{\mu\nu} V_{\star}(\Psi) \right], \\ & D_\mu F^{\mu\nu} = \frac{1}{\sqrt{2}} [\Psi^\alpha, \Psi^\beta]_{\star} (\mathbf{C} \gamma^{(5)} \gamma^\nu)_{\alpha\beta}, \\ & \gamma^\mu D_\mu \Psi - \frac{1}{\sqrt{2}} \gamma^{(5)} \mathbf{C}^{-1} \partial_\Psi \text{Tr} [V_{\star}(\Psi)] = 0. \end{aligned} \quad (3.183)$$

The present paper has dealt with its DFT generalization.

²¹This equivalent form of the equations allows us to omit the explicit writing of the inner Klein operators (3.30) – although they turn out to be useful in order to make explicit contact with Fronsdal equations in the perturbative expansion – and the Weyl zero-form (which can be reconstructed from Ψ).

4 Comments

In this work, we have constructed a higher spin double field theory which is an extension of DFT by the $\mathbf{HS}(4)$ -valued fields present in Vasiliev equations. We have proposed an invariant functional and derived the corresponding Euler-Lagrange equations, in terms of the semi-covariant geometry which manifests all the symmetries. Further, we have identified a minimal set of BPS conditions which automatically solve all the equations of motion. The conditions reduce to the four-dimensional bosonic Vasiliev HS equations, once extra algebraic conditions, (2.55), (2.56) and (2.57) are imposed. By introducing the so-called outer Klein operators and relaxing some parity constraint, it should be possible to extend the analysis to the supersymmetric Vasiliev equations. Besides, without the extra constraints, while employing the “Yang-Mills” potential (2.58), our proposal might provide some bridge between open string and higher spin theory.

The linear dilaton vacuum solution (2.51) derived in section 3.7 is valid for arbitrary values of the DFT cosmological constant, including the trivial case of $\Lambda_{\text{DFT}} = 0$. As can be seen easily from (3.139), the vacuum does not satisfy the extra condition of (2.55). That is to say, the linear dilaton vacuum is a genuine DFT configuration which cannot be realized in the undoubled Vasiliev HS theory. Surely, it differs from the known AdS solution of the Vasiliev equations which utilizes the $\mathfrak{so}(2, 3)$ algebra of (3.43).

While the “deformed oscillator” potential (2.59) seems to be a proper choice of the potential, as the deformed oscillator relations (3.179) solve the algebraic BPS condition (2.48), the alternative “Yang-Mills” potential (2.58) also appears to deserve further investigations: in the low energy limit, we have $\mathcal{Q}_{\pm} \rightarrow m^3$ and hence the deformed oscillator relations can be dynamically achieved. As the higher spin gauge theory is expected to arise in a tensionless limit of string theory, the incorporation of a Brout-Englert-Higgs mechanism may give mass to the higher spin fields and make contact with string field theory.

The spin groups in (supersymmetric) double field theory are known to be twofold which reflects the existing two separate locally inertial frames for the left and the right closed string modes [67]. Yet, in the present work we have focused on one of the two spin groups and extended it to include the higher spin gauge symmetry, $\mathbf{HS}(4)$. It will be of interest to extend the twofold spin groups to realize the ‘doubled’ higher spin algebras *c.f.* [68],

$$\mathbf{Spin}(1, D-1) \times \mathbf{Spin}(D-1, 1) \longrightarrow \mathbf{Spin}(1, D-1) \times \mathbf{HS}(D) \times \mathbf{Spin}(D-1, 1) \times \overline{\mathbf{HS}}(D). \quad (4.1)$$

Instead of the Riemannian parametrization of the DFT-vielbein (3.46), we may consider an ansatz where the upper 4×4 block of V_{Ap} is degenerate, and hence does not admit any Riemannian geometry interpretation [49]. Such a non-Riemannian geometry was shown in [38] to provide a genuine stringy background for the non-relativistic closed string theory *a la* Gomis and Ooguri [69]. In this way, we might be able to obtain a non-relativistic higher spin gravity.

On the one hand, inspired by the conjectured relation between Vasiliev theory and the tensionless limit of *open* strings, we have constructed a HS-DFT in which the $\mathbf{HS}(4)$ -valued fields present in Vasiliev equations are treated as ‘matter’ minimally coupled to DFT, while the DFT is treated as gravity and all the NS-NS massless fields are $\mathbf{HS}(4)$ -singlets. On the other hand, in the light of the relation between higher spin gravity and the tensionless limit of *closed* strings, it would be highly desirable to build a HS-DFT in a much more ambitious sense: a fully unified theory with a single massless spin-two field transforming both under DFT and HS gauge transformations, *i.e.* a HS-DFT *gravity*. Certainly this goes beyond the scope of the present work.

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APPENDIX

A Star product and Wick Theorem

Here, explicitly we present our own proofs for the various properties of the star product.

A.1 Equivalence of (3.21) and (3.23)

The equivalence of the two expressions for the star product, (3.21) and (3.23) can be directly established:

$$\begin{aligned}
& \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ e^{\bar{\lambda}_+\rho_+} f(x, \zeta + \lambda_+, \bar{\zeta}) g(x, \zeta, \bar{\zeta} + \bar{\rho}_+) \\
&= \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ e^{-\bar{\rho}_+\lambda_+} f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(\bar{\rho}_+\alpha \frac{\partial}{\partial \bar{\zeta}_\alpha}\right) g(x, \zeta, \bar{\zeta}) \\
&= \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(-\frac{\partial}{\partial \lambda_+^\alpha} \frac{\partial}{\partial \bar{\zeta}_\alpha}\right) \left[e^{-\bar{\rho}_+\lambda_+} g(x, \zeta, \bar{\zeta})\right] \\
&= \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \lambda_+^\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha}\right) \left[e^{-\bar{\rho}_+\lambda_+} g(x, \zeta, \bar{\zeta})\right] \\
&= \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \zeta_\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha}\right) \left[e^{-\bar{\rho}_+\lambda_+} g(x, \zeta, \bar{\zeta})\right] \tag{A.1} \\
&= \frac{1}{(2\pi)^4} \int d^4\lambda_+ \int d^4\rho_+ e^{\bar{\lambda}_+\rho_+} f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \zeta_\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha}\right) g(x, \zeta, \bar{\zeta}) \\
&= \int d^4\lambda_+ \delta(\lambda_+) f(x, \zeta + \lambda_+, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \zeta_\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha}\right) g(x, \zeta, \bar{\zeta}) \\
&= f(x, \zeta, \bar{\zeta}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \zeta_\alpha} \frac{\overrightarrow{\partial}}{\partial \bar{\zeta}_\alpha}\right) g(x, \zeta, \bar{\zeta}),
\end{aligned}$$

where, to proceed from the third line to the fourth line, integrations by part have been performed.

A.2 Associativity of the star product (3.27)

The star product satisfies the associativity (3.27) as

$$\begin{aligned}
& (f \star g) \star h \\
&= \frac{1}{(2\pi)^4} \int d^4 \lambda_+ \int d^4 \rho_+ e^{\bar{\lambda}_+ \rho_+} [f(\zeta + \lambda_+, \bar{\zeta}) \star g(\zeta + \lambda_+, \bar{\zeta})] h(\zeta, \bar{\zeta} + \bar{\rho}_+) \\
&= \frac{1}{(2\pi)^8} \int d^4 \lambda_+ \int d^4 \rho_+ \int d^4 \lambda'_+ \int d^4 \rho'_+ e^{\bar{\lambda}_+ \rho_+ + \bar{\lambda}'_+ \rho'_+} f(\zeta + \lambda_+ + \lambda'_+, \bar{\zeta}) g(\zeta + \lambda_+, \bar{\zeta} + \bar{\rho}'_+) h(\zeta, \bar{\zeta} + \bar{\rho}_+) \\
&= \frac{1}{(2\pi)^8} \int d^4 \lambda_+ \int d^4 \rho''_+ \int d^4 \lambda''_+ \int d^4 \rho'_+ e^{\bar{\lambda}_+ \rho''_+ + \bar{\lambda}''_+ \rho'_+} f(\zeta + \lambda''_+, \bar{\zeta}) g(\zeta + \lambda_+, \bar{\zeta} + \bar{\rho}'_+) h(\zeta, \bar{\zeta} + \bar{\rho}'_+ + \bar{\rho}''_+) \\
&= \frac{1}{(2\pi)^4} \int d^4 \lambda''_+ \int d^4 \rho'_+ e^{\bar{\lambda}''_+ \rho'_+} f(\zeta + \lambda''_+, \bar{\zeta}) [g(\zeta, \bar{\zeta} + \bar{\rho}'_+) \star h(\zeta, \bar{\zeta} + \bar{\rho}'_+)] \\
&= f \star (g \star h) ,
\end{aligned} \tag{A.2}$$

where we have made the change of variables for the Majorana spinors: $\lambda''_+ = \lambda_+ + \lambda'_+$ and $\rho''_+ = \rho_+ - \rho'_+$.

A.3 Isomorphism, (3.28)

Henceforth we prove the isomorphism (3.28),²²

$$: f(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : = \hat{\mathcal{O}}[f(\zeta, \bar{\zeta}) \star g(\zeta, \bar{\zeta})] . \tag{A.3}$$

Any hatted object is an operator. In particular, the bosonic spinorial coordinates, ζ^α and $\bar{\zeta}_\beta$, are mapped to the operators,

$$\hat{\zeta}^\alpha = \hat{\mathcal{O}}[\zeta^\alpha] , \quad \hat{\bar{\zeta}}_\alpha = \hat{\mathcal{O}}[\bar{\zeta}_\alpha] , \tag{A.4}$$

which satisfy the non-commutative algebra, *c.f.* (2.8),

$$[\hat{\zeta}^\alpha, \hat{\bar{\zeta}}_\beta] = \hat{\zeta}^\alpha \hat{\bar{\zeta}}_\beta - \hat{\bar{\zeta}}_\beta \hat{\zeta}^\alpha = \delta^\alpha_\beta , \quad \hat{\zeta}^\alpha \hat{\zeta}^\beta = \hat{\zeta}^\beta \hat{\zeta}^\alpha , \quad \hat{\bar{\zeta}}_\alpha \hat{\bar{\zeta}}_\beta = \hat{\bar{\zeta}}_\beta \hat{\bar{\zeta}}_\alpha . \tag{A.5}$$

The Wick ordering, denoted by the colon, prescribes to place all the unbarred (annihilation) operators, $\hat{\zeta}^\alpha$, to the right and the barred (creation) operators, $\hat{\bar{\zeta}}_\beta$, to the left. For example,

$$: \hat{\bar{\zeta}}_\beta \hat{\zeta}^\alpha : = \hat{\bar{\zeta}}_\beta \hat{\zeta}^\alpha , \quad : \hat{\zeta}^\alpha \hat{\bar{\zeta}}_\gamma \hat{\zeta}^\beta : = \hat{\bar{\zeta}}_\gamma \hat{\zeta}^\alpha \hat{\zeta}^\beta , \quad : \hat{\bar{\zeta}}_\beta \hat{\zeta}^\alpha \hat{\bar{\zeta}}_\delta \hat{\zeta}^\gamma : = \hat{\bar{\zeta}}_\beta \hat{\bar{\zeta}}_\delta \hat{\zeta}^\alpha \hat{\zeta}^\gamma . \tag{A.6}$$

²²For the alternative Weyl normal ordered star product, see *e.g.* [70, 71].

For an arbitrary function of the internal commuting coordinates, $f(\zeta, \bar{\zeta})$, the corresponding operator, $\hat{\mathcal{O}}[f(\zeta, \bar{\zeta})]$, is defined subject to the Wick ordering prescription,

$$\hat{\mathcal{O}}[f(\zeta, \bar{\zeta})] = : f(\hat{\zeta}, \hat{\bar{\zeta}}) : . \quad (\text{A.7})$$

It is straightforward to verify the following preliminary relations to the isomorphism (3.28),

$$\begin{aligned} \zeta^\alpha \star f(\zeta, \bar{\zeta}) &= \zeta^\alpha f(\zeta, \bar{\zeta}) + \frac{\partial}{\partial \zeta_\alpha} f(x, \zeta, \bar{\zeta}) , & \bar{\zeta}_\alpha \star f(\zeta, \bar{\zeta}) &= \bar{\zeta}_\alpha f(\zeta, \bar{\zeta}) , \\ f(\zeta, \bar{\zeta}) \star \bar{\zeta}_\alpha &= \bar{\zeta}_\alpha f(\zeta, \bar{\zeta}) + \frac{\partial}{\partial \bar{\zeta}^\alpha} f(x, \zeta, \bar{\zeta}) , & f(\zeta, \bar{\zeta}) \star \zeta^\alpha &= \zeta^\alpha f(\zeta, \bar{\zeta}) , \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned} \hat{\zeta}^\alpha \hat{\mathcal{O}}[f(\zeta, \bar{\zeta})] &= \hat{\zeta}^\alpha : f(\hat{\zeta}, \hat{\bar{\zeta}}) : = : \hat{\zeta}^\alpha f(\hat{\zeta}, \hat{\bar{\zeta}}) : + [\hat{\zeta}^\alpha, : f(\hat{\zeta}, \hat{\bar{\zeta}}) :] = \hat{\mathcal{O}}[\zeta^\alpha \star f(\zeta, \bar{\zeta})] , \\ \hat{\bar{\zeta}}_\alpha \hat{\mathcal{O}}[f(\zeta, \bar{\zeta})] &= \hat{\bar{\zeta}}_\alpha : f(\hat{\zeta}, \hat{\bar{\zeta}}) : = : \hat{\bar{\zeta}}_\alpha f(\hat{\zeta}, \hat{\bar{\zeta}}) : = \hat{\mathcal{O}}[\bar{\zeta}_\alpha f(\zeta, \bar{\zeta})] = \hat{\mathcal{O}}[\bar{\zeta}_\alpha \star f(\zeta, \bar{\zeta})] . \end{aligned} \quad (\text{A.9})$$

Now we assume that, the isomorphism (3.28) holds up to the n th order polynomials of ζ and $\bar{\zeta}$, say $f_n(\zeta, \bar{\zeta})$, and an arbitrary function, $g(\zeta, \bar{\zeta})$, *i.e.*

$$: f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : = \hat{\mathcal{O}}[f_n(\zeta, \bar{\zeta}) \star g(\zeta, \bar{\zeta})] . \quad (\text{A.10})$$

The preliminary results (A.9) show that indeed (A.10) holds for $n = 1$. In order to establish a mathematical induction proof, we need to consider $(n+1)$ th order polynomials, or equivalently both $\zeta^\alpha f_n(\zeta, \bar{\zeta})$ and $\bar{\zeta}_\alpha f_n(\zeta, \bar{\zeta})$. Utilizing (A.9), (A.10) and the associativity of the product, we get

$$\begin{aligned} : \hat{\zeta}^\alpha f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : &= : f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : \hat{\zeta}^\alpha : g(\hat{\zeta}, \hat{\bar{\zeta}}) : \\ &= : f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : \hat{\mathcal{O}}[\zeta^\alpha \star g(\zeta, \bar{\zeta})] \\ &= \hat{\mathcal{O}}[f_n(\zeta, \bar{\zeta}) \star \{\zeta^\alpha \star g(\zeta, \bar{\zeta})\}] \\ &= \hat{\mathcal{O}}[\{f_n(\zeta, \bar{\zeta}) \star \zeta^\alpha\} \star g(\zeta, \bar{\zeta})] \\ &= \hat{\mathcal{O}}[\{\zeta^\alpha f_n(\zeta, \bar{\zeta})\} \star g(\zeta, \bar{\zeta})] , \end{aligned} \quad (\text{A.11})$$

and also

$$\begin{aligned}
: \hat{\bar{\zeta}}_\alpha f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : &= \hat{\bar{\zeta}}_\alpha : f_n(\hat{\zeta}, \hat{\bar{\zeta}}) : : g(\hat{\zeta}, \hat{\bar{\zeta}}) : \\
&= \hat{\bar{\zeta}}_\alpha \hat{O}[f_n(\zeta, \bar{\zeta}) \star g(\zeta, \bar{\zeta})] \\
&= \hat{O}[\bar{\zeta}_\alpha \star \{f_n(\zeta, \bar{\zeta}) \star g(\zeta, \bar{\zeta})\}] \\
&= \hat{O}[\{\bar{\zeta}_\alpha \star f_n(\zeta, \bar{\zeta})\} \star g(\zeta, \bar{\zeta})] \\
&= \hat{O}[\{\bar{\zeta}_\alpha f_n(\zeta, \bar{\zeta})\} \star g(\zeta, \bar{\zeta})] .
\end{aligned} \tag{A.12}$$

These two results complete our mathematical induction proof.

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